

# Algebraic Algorithms for $b$ -Matching, Shortest Undirected Paths, and $f$ -Factors\*

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## Abstract

Let  $G = (V, E)$  be a graph with  $f : V \rightarrow \mathbb{Z}_+$  a function assigning degree bounds to vertices. We present the first efficient algebraic algorithm to find an  $f$ -factor. The time is  $O(f(V)^\omega)$ . More generally for graphs with integral edge weights of maximum absolute value  $W$  we find a maximum weight  $f$ -factor in time  $\tilde{O}(Wf(V)^\omega)$ . (The algorithms are randomized, correct with high probability and Las Vegas; the time bound is worst-case.) We also present three specializations of these algorithms: For maximum weight perfect  $f$ -matching the algorithm is considerably simpler (and almost identical to its special case of ordinary weighted matching). For the single-source shortest-path problem in undirected graphs with conservative edge weights, we present a generalization of the shortest-path tree, and we compute it in  $\tilde{O}(Wn^\omega)$  time. For bipartite graphs, we improve the known complexity bounds for vertex capacitated max-flow and min-cost max-flow on a subclass of graphs.

## 1 Introduction

$b$ -matching and  $f$ -factors are basic combinatorial notions that generalize non-bipartite matching, min-cost network flow, and others. This paper presents the first efficient algebraic algorithms for both weighted and unweighted  $b$ -matchings and  $f$ -factors. Our algorithms for this broad class of problems are the most efficient algorithms known for a subclass of instances (graphs of high density, low degree constraints and low edge weights). We also discuss single-source all-sinks shortest paths in conservative undirected graphs. (There is no known reduction to directed graphs.) We prove the existence of a simple shortest-path "tree" for this setting. We also give efficient algorithms – combinatoric for sparse graphs and algebraic for dense – to construct it.

We must first define  $b$ -matching and  $f$ -factors. The literature is inconsistent but in essence we follow the classification of Schrijver [27]. For an undirected multigraph  $G = (V, E)$  with a function  $f : V \rightarrow \mathbb{Z}_+$ , an  $f$ -factor is a subset of edges wherein each vertex  $v \in V$  has degree exactly  $f(v)$ . For an undirected graph  $G = (V, E)$  with a function  $b : V \rightarrow \mathbb{Z}_+$ , a (perfect)  $b$ -matching is a function  $x : E \rightarrow \mathbb{Z}_+$  such that each  $v \in V$  has  $\sum_{w:vw \in E} x(vw) = b(v)$ . The fact that  $b$ -matchings

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have an unlimited number of copies of each edge makes them decidedly simpler. For instance  $b$ -matchings have essentially the same blossom structure (and linear programming dual variables) as ordinary matching [27, Ch.31]. Similarly our algorithm for weighted  $b$ -matching is almost identical to its specialization to ordinary matching ( $b \equiv 1$ ). In contrast the blossoms and dual variables for weighted  $f$ -factors are more involved [27, Ch.32] and our algorithm is more intricate. Thus our terminology reflects the difference in complexity of the two notions.<sup>1</sup>

The paper begins with unweighted  $f$ -factors, i.e., we wish to find an  $f$ -factor or show none exists. Let  $\phi = f(V)$  (or  $b(V)$ ). We extend the Tutte matrix from matching to  $f$ -factors, i.e., we present a  $\phi \times \phi$  matrix that is symbolically nonsingular iff the graph has an  $f$ -factor. Such a matrix can be derived by applying the Tutte matrix to an enlarged version of the given graph, or by specializing Lovász's matrix for matroid parity [18]. But neither approach is compact enough to achieve our time bounds.<sup>2</sup> Then we reuse the elimination framework for maximum cardinality matching, due to Mucha and Sankowski [21] and Harvey [17]. This allows us to find an  $f$ -factor in  $O(\phi^\omega)$  randomized time.<sup>3</sup> For dense graphs and small degree-constraints this improves the best-known time bound of  $O(\sqrt{\phi}m)$  [12], although the latter is deterministic.

Time	Author
$O(n^2B)$	Pulleyblank (1973) [23]
$O(n^2m \log B)$	Marsh (1979) [20]
$O(m^2 \log n \log B)$	Gabow (1983) [12]
$O(n^2m + n \log B(m + n \log n))$	Anstee (1987) [2]
$O(n^2 \log n(m + n \log n))$	Anstee (1987) [2]
$\tilde{O}(W\phi^\omega)$	this paper

**Table 1:** Time bounds for maximum  $b$ -matching.  $B$  denotes  $\max_v b(v)$ .

Complexity	Author
$O(\phi n^3)$	Urquhart (1965) [32]
$O(\phi(m + n \log n))$	Gabow (1983+1990) [12, 13]
$\tilde{O}(W\phi^\omega)$	this paper

**Table 2:** Time bounds for maximum weight  $f$ -factors on simple graphs.

We turn to the more difficult weighted version of the problem. Here every edge has a numeric weight; for complexity results we assume weights are integers of magnitude  $\leq W$ . We seek a *maximum  $f$ -factor*, i.e., an  $f$ -factor with the greatest possible total weight. Efficient algebraic algorithms have been given for maximum matching ( $f \equiv 1$ ) in time  $\tilde{O}(Wn^\omega)$ , first for bipartite graphs [26] and recently for general graphs [7].<sup>4</sup>

<sup>1</sup>Another version of  $b$ -matching considers  $b(v)$  as an upper bound on the desired degree of  $v$ . This easily reduces to weighted perfect  $b$ -matching by taking 2 copies of  $G$  joined by zero-weight edges. On the other hand a *capacitated  $b$ -matching* is defined by giving an upper bound  $u(e)$  to each value  $x(e)$ . The simplicity of the uncapacitated case is lost, and we are back to  $f$ -factors.

<sup>2</sup>The Tutte matrix becomes too large,  $m \times m$ . Lovász's matrix is  $\phi \times \phi$  but can involve integers that are too large, size  $n^n$  or more. Our matrix only involves integers  $\pm 1$ .

<sup>3</sup> $O(n^\omega)$  is the time needed for a straight-line program to multiply two  $n \times n$  matrices; the best-known bound on  $\omega$  is  $< 2.3727$  [33].

<sup>4</sup>The  $\tilde{O}$  notation ignores factors of  $\log(n\phi W)$ .

The usual approach to generalized matching problems is by problem reduction. For instance in [27], Ch.31 proves the properties of the  $b$ -matching linear program and polytope by reducing to ordinary matching via vertex splitting; then Ch.32 reduces  $f$ -factors (capacitated  $b$ -matching) to  $b$ -matching. Efficient algorithms also use vertex splitting [12] or reduction to the bipartite case (plus by further processing) [2]. But reductions may obscure some structure. To avoid this our algorithms use a direct approach, and we get the following rewards. For  $b$ -matching, as mentioned, the similarity of blossoms to ordinary matching blossoms leads to an algorithm that is no more involved than ordinary matching. For undirected shortest paths we get a simple definition of a generalized shortest-path tree. (Again such a definition may have been overlooked due to reliance on reductions, see below.) For  $f$ -factors we get a detailed understanding of the more complicated versions of the structures that first emerge in  $b$ -matchings (2-edge connected components giving the cyclic part of blossoms – see Section 8.1) and in shortest paths (bridges giving the incident edges of blossoms – these correspond to the (ungeneralized) shortest-path tree – see Section 8.2).

All three of our non-bipartite algorithms are implementations of the "shrinking procedure" given in [14] (a variant is the basis of the weighted matching algorithm of [7]). This procedure gives a direct way to find the optimum blossoms for a weighted  $f$ -factor – simply put, each blossom is (a subgraph of) a maximum weight " $2f$ -unifactor" (a type of  $2f$ -factor) in the graph with (the cyclic part of) all heavier blossoms contracted (see [14] or Section 7). Note that the classic weighted matching algorithm of Edmonds [8] finds the optimum blossoms, but only after forming and discarding various other blossoms. So this approach does not provide a direct definition of the optimum blossoms.

The first step of our algorithms use our generalized Tutte matrix to find the optimum dual variables of the vertices. Then we execute the shrinking procedure to get the blossoms, their duals, and a "weighted blossom tree" that gives the structure of the optimum  $f$ -factor. (This step is combinatoric. It is based on the detailed structure of  $2f$ -unifactors that we derive.) The last step finds the desired  $f$ -factor using a top-down traversal of the weighted blossom tree: At each node we find an  $f$ -factor of a corresponding graph, using our algorithm for unweighted  $f$ -factors. In summary our algorithms (like [7] for ordinary matching) can be viewed as a (combinatoric) reduction of the weighted  $f$ -factor problem into two subproblems: finding the optimum dual variables of the vertices, and finding an unweighted  $f$ -factor.

To facilitate understanding of the general  $f$ -factor algorithm we begin by presenting its specialization to two subcases. First  $b$ -matching. The blossoms, and hence the  $2b$ -unifactors, differ little from ordinary graph matching. As a result our development for weighted  $b$ -matching is essentially identical the special case of ordinary matching, in terms of both the underlying combinatorics and the algorithmic details. When specialized to ordinary matching our algorithm provides a simple alternative to [7]. In fact an advantage is that our algorithm is Las Vegas – the dual variables allow us to check if the  $b$ -matching is truly optimum. (Our approach to weighted matching/ $b$ -matching differs from [7] – at the highest level, we work with critical graphs while [7] works with perfect graphs.)

Next we discuss shortest paths in undirected graphs with a conservative weight function – negative edges are allowed but not negative cycles. The obvious reduction to a directed graph (replace undirected edge  $uv$  by directed edges  $uv, vu$ ) introduces negative cycles, and it is unclear how to handle this problem by the usual shortest-path techniques.

We consider the single-source all-sinks version of the problem. Again, this problem is often solved by reduction, first to the single-source single-sink version and then to perfect matching, using either T-joins [27, pp.485–486] or vertex-splitting [27, p.487]. A path can be viewed as a type of 2-factor.

(For instance an  $ab$ -path is an  $f$ -factor if we enlarge  $G$  with a loop at every vertex  $v \in V$  and set  $f(v) = 2$  for  $v \in V - \{a, b\}$ ,  $f(a) = f(b) = 1$ .) This enables us to solve the all-sinks version directly. Examining the blossom structure enables us to define a generalized shortest-path tree that, like the standard shortest-path tree for directed graphs, specifies a shortest path to every vertex from a chosen source. It is a combination of the standard shortest-path tree and the blossom tree. We give a complete derivation of the existence of this shortest-path structure, as well as an algebraic algorithm to construct it in time  $\tilde{O}(Wn^\omega)$ . We also construct the structure with combinatoric algorithms, in time  $O(n(m + n \log n))$  or  $O(\sqrt{n\alpha(m, n)} \log n \cdot m \log(nW))$ . These bounds are all within logarithmic factors of the best-known bounds for constructing the directed shortest-path tree [27, Ch. 8], [34, 25].

Although the shortest-path problem is classic, our definition of this structure appears to be new. Most notably, Sebö has characterized the structure of single-source shortest paths in undirected graphs, first for graphs with  $\pm 1$  edge weights [29] and then extending to general weights by reduction [30]. Equation (4.2) of [29] (for  $\pm 1$ -weights, plus its version achieved by reduction for arbitrary weights) characterizes the shortest paths from a fixed source in terms of how they enter and leave "level sets" determined by the distance function. [29] also shows that the distances from the source can be computed using  $O(n)$  perfect matching computations. Our structure differs from [29, 30]: it does not give a necessary and sufficient condition to be a shortest path, but it gives an exact specification of a specific set of shortest paths that are simply related to one another (as in the standard shortest-path tree). Note that one can give an alternative proof of the existence of our structure by starting from the results of [29, 30].

The general algorithm for maximum  $f$ -factors is the most difficult part of the paper. It involves a detailed study of the properties of blossoms. A simple example of how these blossoms differ from ordinary matching is that the hallmark of Edmonds' blossom algorithm – "blossoms shrink" – is not quite true. In other words for ordinary matching a blossom can be contracted and it becomes just an ordinary vertex. For  $f$ -factors we can contract the "cycle" part of the blossom, but its incident edges remain in the graph and must be treated differently from ordinary edges (see Section 8.2). Our discussion of shortest paths introduces this difficulty in the simplest case – here a blossom has exactly 1 incident edge (as opposed to an arbitrary number). Even ignoring this issue, another difficulty is that there are three types of edges that behave differently (see [14], or Lemmas 33 and 35) and the type of an edge is unknown to the algorithm! Again the three types are seen to arise naturally in shortest paths. Our contribution is to develop the combinatoric properties of these edges and blossoms so the shrinking procedure can be executed efficiently, given only the information provided by the Tutte matrix in our algebraic algorithm.

While non-bipartite graphs present the greatest technical challenge, we also achieve some best-known time bounds for two bipartite problems, maximum network flow and min-cost network flow. Bipartite  $f$ -factors generalize network flow: max-flow (min-cost max-flow) is a special case of unweighted (weighted) bipartite  $f$ -factors, respectively e.g. [15]. The question of an efficient algebraic max-flow algorithm has confronted the community for some time. The only advance is the algorithm of Cheung et. al. [5], which checks whether  $d$  units of flow can be sent across a unit-capacity network in  $O(d^{\omega-1}m)$  time. We consider networks with integral vertex and edge capacities bounded by  $D$ . We find a max-flow in time  $\tilde{O}((Dn)^\omega)$  time and a min-cost max-flow in  $\tilde{O}(W(Dn)^\omega)$  time. The latter algorithm handles convex edge cost functions (with integral break-points) in the same time bound. The max-flow problem has a rich history (see e.g. [27, Chs. 10, 12]) and our time bounds are the best-known for dense graphs with moderately high vertex capacities. Specifically, previous

	Section					
	5	8	9	10	12	13
<b>Shortest paths</b>						
find distances				C		X
find forest		C				
extract sp-tree			C			
<b><i>b</i>-matching</b>						
find duals						X
find forest & extract	C					
unweighted <i>b</i> -matching					X	
<b><i>f</i>-factors</b>						
find duals						X
find forest & extract		C				
unweighted <i>f</i> -factor					X	

**Table 3:** Sections for each step of the algorithm. X is an algebraic algorithm, C is combinatoric. Shortest-path distances can be found algebraically or combinatorially.

algorithms for vertex-capacitated max-flow in dense networks (i.e.,  $m = \Theta(n^2)$ ) use  $O(n^3/\log n)$  time [4] or  $O(n^{8/3} \log D)$  time [16]. Previous algorithms for dense graph min-cost max-flow use  $O(n^3 \log D)$  time [10] or  $O(n^3 \log n)$  time [22]. Previous algorithms for minimum convex-cost max-flow use  $O(Dn^3 \log D)$  time (by simple reduction to min-cost max-flow) or in  $O(n^3 \log D \log(nW))$  time [15].

In summary the novel aspects of this paper are:

- new time bounds for the fundamental problems of *b*-matching, undirected single-source shortest paths, and *f*-factors;
- extension of the Tutte matrix for matching to *f*-factors;
- definition of the shortest-path structure for undirected graphs, plus algebraic and combinatorial algorithms to construct it;
- an algebraic algorithm for *b*-matching that is no more involved than ordinary matching;
- an algebraic algorithm for *f*-factors based on new combinatorial properties of blossoms, which are qualitatively different from matching blossoms;
- new time bounds for vertex-capacitated max-flow, min-cost max-flow and minimum convex-cost max-flow on dense graphs.

**Organization of the paper** The next two sections define our terminology and review the algebraic tools that we use.

Section 4 gives an overview of the entire paper, by discussing the special case of bipartite *f*-factors. (The reader should bear in mind that the non-bipartite case is the highlight of this paper. It is technically much more demanding.) In detail, Section 4 starts with an  $O(\phi^\omega)$  time algorithm for unweighted bipartite *f*-factors. This requires our generalized Tutte matrix (Section 4.1; Section 11 extends this to non-bipartite graphs) plus the algorithmic details (Section 4.2; Section 12 generalizes these details to non-bipartite graphs). Section 4.3 gives the algorithm for weighted bipartite *f*-factors. Flows are discussed in Section 4.4.

As mentioned above the algorithms for general graphs have three steps:

- (i) find the weights of factors of perturbed graphs;
- (ii) use the weights to construct a blossom forest;
- (iii) use an unweighted algorithm to extract an optimal solution from the blossom forest.

These steps are implemented in different sections, depending on the problem of interest, as indicated in Table 3. For instance the complete  $b$ -matching algorithm starts with the algorithm of Section 13 to get the dual variables; then it constructs a weighted blossom forest and traverses the forest to extract the optimal solution, using algorithms in Section 5; the traversal uses the algorithm of Section 12 to find various unweighted  $b$ -matchings.

The most involved part of the paper is the construction of blossom forest for  $f$ -factors in Section 8. This section is preceded by two sections that introduce our combinatoric ideas in simpler settings: Section 5 gives the combinatoric portion of our weighted  $b$ -matching algorithm. Section 6 proves the existence of our shortest-path structure. Section 7 reviews fundamental background material on  $f$ -factors (illustrating it by shortest-paths). After the combinatoric algorithm for weighted  $f$ -factors in Section 8, Section 9 gives the algorithm to construct the shortest-path structure, and Section 10 gives the combinatorial algorithms for shortest-path weights. Sections 11–12 show how to find unweighted  $f$ -factors in general graphs. Section 13 shows how to compute the weights of perturbations of  $f$ -factors. Finally Section 14 concludes by posing several new open problems.

Given the length of this submission, we remark that various portions can be read independently. Section 4 gives the whole development in the bipartite case. Section 5 gives the combinatoric part of the  $b$ -matching algorithm starting from first principles. Section 6 derives the shortest-path structure from first principles, entirely in the language of shortest paths rather than matching. The related material in Section 9 is itself independent, given the definition of the shortest-path structure (Section 6.1). Alternatively Sections 5–6 can be skipped to go directly to the combinatoric part of the  $f$ -factor algorithm (Sections 7–8).

## 2 Problem definitions

The symmetric difference of sets is denoted by  $\oplus$ , i.e.,  $A \oplus B = (A - B) \cup (B - A)$ . We use a common convention to sum function values: If  $f$  is a real-valued function on elements and  $S$  is a set of such elements,  $f(S)$  denotes  $\sum\{f(v) : v \in S\}$ . Similarly if  $z$  is a function on sets of elements then  $z\{S : S \in \mathcal{S}\}$  denotes  $\sum\{z(S) : S \in \mathcal{S}\}$ .

Let  $G = (V, E)$  be an undirected graph, with vertex set  $V = \{1, \dots, n\}$ . We sometimes write  $V(G)$  or  $E(G)$  to denote vertices or edges of graph  $G$ . A *walk* is a sequence  $A = v_0, e_1, v_1, \dots, e_k, v_k$  for vertices  $v_i$  and edges  $e_i = v_{i-1}v_i$ . The notation  $v_0v_k$ -*walk* provides the two endpoints. The *length* of  $A$  is  $k$ , and the parity of  $k$  makes  $A$  *even* or *odd*. A *trail* is an edge-simple walk. A *circuit* is a trail that starts and ends at the same vertex. The vertex-simple analogs are *path* and *cycle*.

In an undirected multigraph  $G = (V, E)$  each edge  $e \in E$  has a positive multiplicity  $\mu(e)$ . Each copy of a fixed  $e \in E$  is a distinct edge, e.g., a trail may up to  $\mu(e)$  distinct copies of  $e$ . For a multiset  $S$ ,  $2S$  denotes  $S$  with every multiplicity doubled. Similarly  $2G$  denotes the multigraph  $(V, 2E)$ . If every multiplicity of  $S$  is even then  $S/2$  denotes  $S$  with every multiplicity halved.

For a set of vertices  $S \subseteq V$  and a subgraph  $H$  of  $G$ ,  $\delta(S, H)$  ( $\gamma(S, H)$ ) denotes the set of edges with exactly one (respectively two) endpoints in  $S$  (loops are in  $\gamma$  but not  $\delta$ ).  $d(v, H)$  denotes the

degree of vertex  $v$  in  $H$ . When referring to the given graph  $G$  we often omit the last argument and write, e.g.,  $\delta(S)$ .

An edge weight function  $w$  assigns a numeric weight to each edge. For complexity bounds we assume the range of  $w$  is  $[-W..W]$ , i.e., the set of integers of magnitude  $\leq W$ . The *weight of edge set*  $F \subseteq E$  is  $w(F) = \sum_{e \in F} w(e)$ .  $w$  is *conservative* if there are no negative weight cycles. For multigraphs we sometimes write  $w(e, k)$  to denote the weight of the  $k$ th copy of edge  $e$ .

Let  $G = (V, E)$  be a multigraph. For a function  $f : V \rightarrow \mathbb{Z}_+$ , an *f-factor* is a subset of edges  $F \subseteq E$  such that  $d(v, F) = f(v)$  for every  $v \in V$ . Let  $G = (V, E)$  be a graph, where  $E$  may contain loops  $vv$  but no parallel edges. For a function  $b : V \rightarrow \mathbb{Z}^+$ , a (perfect) *b-matching* is a function  $x : E \rightarrow \mathbb{Z}_+$  such that  $\sum_{w:vw \in E} x(vw) = b(v)$ . A *maximum f-factor* is an *f-factor*  $F$  with maximum weight  $w(F)$ . Similarly a *maximum b-matching* is a perfect *b-matching* of maximum weight.

To simplify the time bounds we assume matrix multiplication time is  $\Omega(n^2 \log n)$ . This allows us to include terms like  $O(m)$  and  $O(m \log n)$  within our overall bound  $O(\phi^\omega)$ . Observe that  $m = O(n^2) = O(\phi^2)$  if there are no parallel edges, or if all copies of an edge have the same weight. In the most general case – arbitrary parallel edges – we can assume  $m = O(n\phi) = O(\phi^2)$  after linear-time preprocessing. In proof, for each edge  $uv$  the preprocessing discards all but the  $f(u)$  largest copies. This leaves  $\leq \sum \{d(u)f(u) : u \in V\} \leq \sum \{nf(u) : u \in V\} \leq n\phi$  edges in  $G$ .

### 3 Algebraic preliminaries

One of the fundamental ideas of our algorithms is to encode graph problems in matrices, in such a way that determinant of a matrix is (symbolically) non-zero if and only if the problem has a solution. The Schwartz-Zippel Lemma [35, 28] provides us with an efficient non-zero test for such symbolic determinants. For our purposes the following simplified version of it suffices.

**Corollary 1** (Schwartz-Zippel). *For any prime  $p$ , if a (non-zero) multivariate polynomial of degree  $d$  over  $\mathbb{Z}_p$  is evaluated at a random point, the probability of false zero is  $\leq d/p$ .*

In order to use this lemma, we will choose primes  $p$  of size  $\Theta(n^c)$  for some constant  $c$ . We note that in a RAM machine with word size  $\Theta(\log n)$ , arithmetic modulo  $p$  can be realized in constant time.

However, finding the right encoding is just the first and the easiest step, whereas the more complicated part is to extract the actual solution from this encoding. In order to do it we use the following algebraic tools. The first one allows us to update the inverse of the matrix after we have changed the matrix itself.

**Lemma 2.** (SHERMAN-MORRISON-WOODBURY FORMULA). *Let  $A$  be  $n \times n$  non-singular matrix, and  $U, V$  be  $n \times k$  matrices, then*

- $A + UV^T$  is non-singular if and only if the  $k \times k$  matrix  $I_k + V^T A^{-1} U$  is non-singular and  $\det(A + UV^T) = \det(A) \det(I_k + V^T A^{-1} U)$ ,
- if  $A + UV^T$  is non-singular then  $(A + UV^T)^{-1} = A^{-1} - A^{-1} U (I_k + V^T A^{-1} U)^{-1} V^T A^{-1}$ .

When  $k = 1$  the matrices  $U$  and  $V$  become length  $n$  vectors. Such an update is called *rank-one update*. In this special case the above lemma is called Sherman-Morrison formula. Observe that for  $k = 1$  we can compute  $(A + UV^T)^{-1}$  from  $A^{-1}$  in  $O(n^2)$  arithmetic operations.

In our algorithms we use the above formula but restricted to submatrices. Let  $R$  ( $C$ ) denote set of rows (respectively columns) of matrix  $A$ . We denote the submatrix of  $A$  restricted to rows

in  $R$  and columns in  $C$  by  $A[R, C]$ . Harvey [17, Corollary 2.1] has observed that  $A^{-1}[S, S]$  can be computed in  $O(|S|^\omega)$  time after updates have been made to submatrix  $A[S, S]$ .

The final tool is rather recent result in symbolic computation by Storjohann [31]. He has shown how to compute a determinant of a polynomial matrix, as well as, how to solve a rational system for polynomial matrix.

**Theorem 3** (Storjohann '03). *Let  $K$  be an arbitrary field,  $A \in K[y]^{n \times n}$  a polynomial matrix of degree  $W$ , and  $b \in K[y]^{n \times 1}$  a polynomial vector of the same degree. Then*

- *rational system solution  $A^{-1}b$  (Algorithm 5 [31]),*
- *determinant  $\det(A)$  (Algorithm 12 [31]),*

*can be computed in  $\tilde{O}(Wn^\omega)$  operations in  $K$ , w.h.p.*

## 4 Outline of the paper, and the bipartite case

This section has two purposes. First it presents our algorithms for bipartite graphs, a simplification of the general approach. (But even in the bipartite case our techniques were not previously known.) Second, it gives a guide to the entire paper: Each time we introduce a construction we discuss how it can be extended to the general case. The section ends by giving our algorithms for vertex-capacitated flow.

### 4.1 Determinant formulations

Consider a simple bipartite graph  $G$ , with both vertex sets  $V_0, V_1$  numbered from 1 to  $n$ . Let  $\phi = \sum_i f(i)/2$ . Define a  $\phi \times \phi$  matrix  $B(G)$ , the *symbolic adjacency matrix of  $G$* ,  $f$ , as follows. A vertex  $i \in V_0$  is associated with  $f(i)$  rows, which are indexed by a pair  $i, r$ , for  $0 \leq r < f(i)$ . Similarly  $j \in V_1$  is associated with  $f(j)$  columns indexed by  $j, c$ , for  $0 \leq c < f(j)$ .  $B(G)$  uses indeterminates  $x_r^{ij}$  and  $y_c^{ij}$  and is defined by

$$B(G)_{i,r,j,c} = \begin{cases} x_r^{ij} y_c^{ij} & ij \in E, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Observe that each edge in the graph is represented by a rank-one submatrix given by the product of two vectors  $x^{ij}(y^{ij})^T$ . Before we prove the main theorems we make the following observation that will show each edge can be used only once.

**Lemma 4.** *Let  $A$  be a symbolic  $n \times n$  matrix, let  $R$  be the set of  $m$  rows, and  $C$  be the set of  $m$  columns of  $A$ . If  $A[R, C]$  has rank bounded by  $r$  then each term in the expansion of  $\det(A)$  contains at most  $r$  elements from  $A[R, C]$ .*

*Proof.* Using Laplace expansion we expand  $\det(A)$  into  $m \times m$  minors that contain all  $m$  rows of  $R$ , i.e.,

$$\det(A) = \sum_{M \subseteq V_1, |M|=m} \text{sgn}(M) \det(A[R, M]) \det(A[V_0 - R, V_1 - M]),$$

where  $\text{sgn}(M) = \prod_{c \in M} (-1)^c$ . Consider now each element of the above sum separately. If  $M$  contains  $> r$  columns of  $C$  then  $\det(A[R, M]) = 0$ , so the elements contributing to  $\det(A)$  have  $\leq r$  columns of  $C$ . Moreover,  $A[V_0 - R, V_1 - M]$  has no rows of  $A[R, C]$ , so  $\det(A[R, M]) \det(A[V_0 - R, V_1 - M])$  has  $\leq r$  entries from  $A[R, C]$ . ■

The determinant of  $B(G)$  is the sum of many different terms each containing exactly  $\phi$  occurrences of variable pairs  $x_r^{ij} y_c^{ij}$ . Each pair  $x_r^{ij} y_c^{ij}$  corresponds to an edge  $ij \in E$ . For a term  $\sigma$  let  $F_\sigma$  denote the multiset of edges that correspond to the variable pairs in  $\sigma$ . Define  $\mathcal{F}$  to be the function that maps each term  $\sigma$  to  $F_\sigma$ .

**Theorem 5.** *Let  $G$  be a simple bipartite graph. The function  $\mathcal{F}$  from terms in  $\det(B(G))$  is a surjection onto the  $f$ -factors of  $G$ . Consequently,  $G$  has an  $f$ -factor if and only if  $\det(B(G)) \neq 0$ .<sup>5</sup>*

*Proof.* First we show that the image of  $\mathcal{F}$  contains all  $f$ -factors of  $G$ . Suppose  $F$  is an  $f$ -factor in  $G$ . Order the edges of  $F$  that are incident to each vertex arbitrarily. If  $ij \in F$  is the  $r + 1$ st edge at  $i$  and the  $c + 1$ st edge at  $j$  then it corresponds to entry  $B(G)_{i,r,j,c}$ . Thus  $F$  corresponds to a nonzero term  $\sigma$  in the expansion of  $\det(B(G))$ . Observe that entries  $B(G)_{i,r,j,c}$  define  $\sigma$  in a unique way, so no other term has exactly the same indeterminates. Of course there can be many terms representing  $F$ .

Can  $\det(B(G))$  contain terms that do not correspond to  $f$ -factors? We show the answer is no. Suppose  $\det(B(G)) \neq 0$  and take any term  $\sigma$  in the expansion of  $\det(B(G))$ .  $\sigma$  corresponds to an  $f$ -factor unless more than one entry corresponds to the same edge of  $G$ . This is impossible because edges are represented by rank-one submatrices and Lemma 4 shows elements of such submatrices appear at most once. ■

Now let  $G$  be a bipartite multigraph. Let  $\mu(e)$  denote the multiplicity of any edge  $e \in E$ . Redefine the corresponding entry in  $B(G)$  by

$$B(G)_{i,r,j,c} = \sum_{k=1}^{\mu(ij)} x_r^{ij,k} y_c^{ij,k}. \quad (2)$$

In other words, now edge  $ij$  of multiplicity  $\mu(ij)$  is represented by a submatrix of rank  $\mu(ij)$ . Hence Lemma 4 shows edge  $ij$  can appear in a term of  $\det(B(G))$  at most  $\mu(ij)$  times. This leads to the following generalization of Theorem 5.

**Corollary 6.** *Let  $G$  be a bipartite multigraph. The function  $\mathcal{F}$  from terms in  $\det(B(G))$  is a surjection onto the  $f$ -factors in  $G$ . Consequently,  $G$  has an  $f$ -factor iff  $\det(B(G)) \neq 0$ .*

The final point of this section concerns the complexity of using the  $B(G)$  matrix. As in most algebraic algorithms we evaluate  $B(G)$  using a random value for each indeterminate to get a matrix  $B$ . If  $G$  is a simple graph this is easily done in time  $O(\phi^2)$ . But the situation is less clear for multigraphs. The most extreme case is exemplified by  $b$ -matching. Consider an arbitrary edge  $ij$  and let  $B[I, J]$  denote the  $f(i) \times f(j)$  submatrix of  $B$  that represents it.  $ij$  has multiplicity  $\mu(ij) \leq \min\{f(i), f(j)\}$ . When  $f(i)$  and  $f(j)$  are  $\Theta(\phi)$  the time to compute the expression of (2) is  $\Theta(\phi)$ . So computing the  $\Theta(\phi^2)$  entries of  $B[I, J]$  uses time  $\Theta(\phi^3)$ . But this can be avoided and we can construct  $B$  in time  $O(\phi^\omega)$ , as follows.

As observed above  $B[I, J]$  is the product of an  $f(i) \times \mu(ij)$  matrix  $X$  of indeterminates  $x_r^{ij,k}$  and an  $\mu(ij) \times f(j)$  matrix  $Y$  of indeterminates  $y_c^{ij,k}$ . Wlog assume  $f(i) \leq f(j)$ . We have the trivial bound  $\mu(ij) \leq f(i)$ . Break up the product  $XY$  into products of  $\mu(ij) \times \mu(ij)$  matrices, i.e., break

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<sup>5</sup>The second part of the theorem suffices for undirected  $f$ -factors. But we will need the stronger claim of surjectivity for the weighted case.

$X$  into  $f(i)/\mu(ij)$  matrices of size  $\mu(ij) \times \mu(ij)$ , and similarly  $Y$ , to get  $XY$  as a sum of  $\frac{f(i)}{\mu(ij)} \frac{f(j)}{\mu(ij)}$  products of  $\mu(ij) \times \mu(ij)$  matrices. Using fast matrix multiplication on these products, the total time to compute  $B$  is bounded by a constant times

$$\sum_{f(i) \leq f(j)} f(i)f(j)\mu(ij)^{\omega-2} \leq \sum_{f(i) \leq f(j)} f(i)^{\omega-1}f(j) \leq \sum_i f(i)^{\omega-1} \left( \sum_{f(i) \leq f(j)} f(j) \right) \leq \sum_i \phi^{\omega-2} f(i) \phi \leq \phi^\omega.$$

**Generalizations** The generalization of these ideas to non-bipartite graphs is given in Section 11. We combine the above idea, submatrices with bounded rank, with the idea of Tutte to construct a skew-symmetric matrix. Additionally we need to take care of self-loops in multigraphs. In the Tutte matrix self-loops do not need appear since they cannot be used in a 1-factor.

## 4.2 Finding $f$ -factors

This section gives our algorithm to find an  $f$ -factor of a bipartite multigraph. We follow the development from [21]: We start with an  $O(\phi^3)$ -time algorithm. Then we show it can be implemented in  $O(\phi^\omega)$  time using the Gaussian elimination algorithm of Bunch and Hopcroft [3].

An *allowed edge* is an edge belonging to some  $f$ -factor. For perfect matchings the notion of allowed edge is easily expressible using the inverse of  $B(G)$ :  $ij$  is allowed if and only if  $B(G)_{i,j}^{-1}$  is non-zero [24]. We will prove a similar statement for bipartite  $f$ -factors. (But such a statement fails for non-bipartite graphs – see Appendix A.) For a given  $f$  define  $f_{i,j}$  to be

$$f_{i,j}(v) = \begin{cases} f(v) - 1 & \text{if } v = i \text{ or } v = j, \\ f(v) & \text{otherwise.} \end{cases}$$

**Lemma 7.** *Let  $G$  be a bipartite multigraph having an  $f$ -factor. Edge  $ij \in E$  is allowed if and only if  $G$  has an  $f_{i,j}$ -factor.*

*Proof.* The “only if” direction is clear: If  $F$  is an  $f$ -factor containing  $ij$ , then  $F - ij$  is an  $f_{i,j}$ -factor.

Conversely, suppose  $F$  does not contain the chosen edge  $ij$ . Take an  $f_{i,j}$ -factor  $F'$  that maximizes  $|F' \cap F|$ .  $F' \oplus F$  contains an alternating  $ij$ -trail  $P$  that starts and ends with edges of  $F$ . In fact  $P$  is a path. (Any cycle  $C$  in  $P$  has even length and so is alternating. This makes  $F' \oplus C$  an  $f_{ij}$ -factor containing more edges of  $F$  than  $F'$ , impossible.)

$ij$  is not the first edge of  $P$  ( $ij \notin F$ ). So  $ij \notin P$ , since  $P$  is vertex simple. Thus  $(F \oplus P) + ij$  is an  $f$ -factor containing  $ij$ . ■

**Lemma 8.** *Let  $G$  be a bipartite multigraph having an  $f$ -factor, let  $B(G)$  be its symbolic adjacency matrix, and let  $i \in V_0$  and  $j \in V_1$ . Then  $(B(G)^{-1})_{j,0,i,0} \neq 0$  if and only if  $G$  has  $f_{i,j}$ -factor.*

*Proof.* Observe that

$$(B(G)^{-1})_{j,0,i,0} = \frac{(-1)^{n(j,0)+n(i,0)} \det(B(G)^{i,0,j,0})}{\det(B(G))},$$

where  $B(G)^{i,0,j,0}$  is the matrix  $B(G)$  with  $i, 0$ 'th row and  $j, 0$ 'th column removed, and  $n(i, k)$  is the actual index of the row or column given by the pair  $i, k$ . We have  $\det(B(G)) \neq 0$  since  $G$  has an  $f$ -factor. Hence  $(B(G)^{-1})_{j,0,i,0} \neq 0$  if and only if  $\det(B(G)^{i,0,j,0}) \neq 0$ . Furthermore  $B(G)^{i,0,j,0}$  is the symbolic adjacency matrix obtained from  $G$  for  $f_{i,j}$ -factors. ■

Observe that by the symmetry of the matrix  $B(G)$ , when  $(B(G)^{-1})_{j,0,i,0} \neq 0$  then as well  $(B(G)^{-1})_{j,\kappa,i,\iota} \neq 0$  for all  $0 \leq \iota < f(i)$  and  $0 \leq \kappa < f(j)$ . Combining the above two lemmas with this observation we obtain the following.

**Corollary 9.** *Let  $G$  be a bipartite multigraph having an  $f$ -factor, and let  $B(G)$  be its symbolic adjacency matrix. The edge  $ij \in E$  is allowed if and only if  $(B(G)^{-1})_{j,\kappa,i,\iota} \neq 0$  for any  $0 \leq \iota < f(i)$  and  $0 \leq \kappa < f(j)$ .*

Being equipped with a tool for finding allowed edges we can now use the Gaussian elimination framework from [21]. The following observation is useful.

**Lemma 10** ([21]). *Let  $A$  be a non-singular  $\phi \times \phi$  matrix and let  $1 \leq i, j \leq n$  be such that  $(A^{-1})_{j,i} \neq 0$ . Let  $A'$  be the matrix obtained from  $A^{-1}$  by eliminating row  $j$  and column  $i$  using Gaussian elimination. Then  $A' = (A^{i,j})^{-1}$  (i.e.,  $A'$  is the Schur complement of  $(A^{-1})^{j,i}$ ).*

The above lemma can be used to obtain the following algorithm that finds an  $f$ -factor.

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**Algorithm 1** An  $O(\phi^3)$  time algorithm to find an  $f$ -factor in a bipartite multigraph  $G$ .

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1: Let  $B(G)$  be the  $\phi \times \phi$  matrix representing  $G, f$ 
2: Replace the variables in  $B(G)$  by random values from  $\mathcal{Z}_p$  for prime  $p = \Theta(\phi^2)$  to obtain  $B$ 
3: If  $B$  is singular return "no  $f$ -factor"
4: (with probability  $\geq 1 - \frac{1}{\phi}$  matrix  $B$  is non-singular when  $B(G)$  is non-singular)    ▷ by Cor. 1
5:  $F := \emptyset$ 
6: Compute  $N := B^{-1}$ 
7: (each column  $i, \iota$  of  $B^{-1}$  has an allowed edge, since  $BB^{-1} = I$  gives  $j, \kappa$  with  $B_{i,\iota,j,\kappa}B_{j,\kappa,i,\iota}^{-1} \neq 0$ )
8: for  $i = [1..n]$  do
9:   for  $\iota = [0..f(i) - 1]$  do
10:    Find  $j, \kappa$  such that  $ij \in E - F$  and  $N_{j,\kappa,i,\iota} \neq 0$     ▷ by Corollary 9 edge  $ij$  is allowed
11:    Eliminate the  $j, \kappa$ 'th row and the  $i, \iota$ 'th column from  $N$     ▷ using Gaussian elimination
12:    (Lemma 10 shows  $N = (B^{i,\iota,j,\kappa})^{-1}$ , i.e.,  $N$  encodes  $f_{i,j}$ -factors, but see below)
13:    Set  $F := F + ij$ 
14:   end for
15: end for
16: Return  $F$ 

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The comment of line 12 is adequate for  $\iota = 0$ . However  $\iota > 0$  requires an additional observation. To see this first recall the logic of each iteration: Let  $f'$  be the residual degree requirement function, i.e., the current  $F$ , enlarged with an  $f'$ -factor of the current graph, gives an  $f$ -factor of  $G$ . In line 10, the  $f'$ -factor  $F'$  that contains  $ij$  is a subgraph of the graph corresponding to (the current)  $N$  and its corresponding matrix  $B$ . Now suppose the iteration for  $\iota = 0$  adds edge  $ip$  to  $F$ . When the row and column for  $ip$  are deleted from  $B$ , the remaining rows for vertex  $i$  still contain entries corresponding to edge  $ip$  (recall the definition of  $B(G)$ ). So when the iteration for  $\iota = 1$  chooses its edge  $ij$ , the corresponding  $f'$ -factor  $F'$  may contain edge  $ip$ . But  $F$  cannot be enlarged with  $F'$ , since that introduces two copies of  $ip$ . The same restriction applies to iterations for  $\iota > 1$ , but now it concerns all previously chosen edges  $ip$ .

Actually there is no problem because we can guarantee an  $f'$ -factor avoiding all the previous  $ip$ 's exists. The guarantee is given by the following corollary to Lemma 7. (Note when  $r = 0$  the corollary is a special case of the lemma. Also, the converse of the corollary holds trivially.)

**Corollary 11.** *Consider a set of edges  $P = \{ip_1, \dots, ip_r\}$ ,  $r \geq 0$ . Suppose  $G - P$  has an  $f$ -factor. If  $G$  has an  $f_{ij}$ -factor for some edge  $ij \notin P$  then  $G - P$  has an  $f$ -factor containing  $ij$ .*

*Proof.* The proof of Lemma 7 applies, assuming we start by taking  $F$  to be the assumed  $f$ -factor. ■

Finally observe that Algorithm 1 is implementing Gaussian elimination on  $B^{-1}$ , the only difference being that pivot elements are chosen to correspond to edges of the graph. If there exists an  $f$ -factor, there is an allowed edge incident to each vertex. Hence, even with this additional requirement Gaussian elimination is able to find a non-zero element in each row of  $B^{-1}$ .

Bunch and Hopcroft [3] show how to speed up the running time of Gaussian elimination from  $O(\phi^3)$  to  $O(\phi^\omega)$ , by using lazy updates to the matrix. Let us divide the columns of the matrix into two almost equal parts. Let  $L$  denote the first  $\lceil \phi/2 \rceil$  columns that are to be eliminated, whereas let  $R$  denote the remaining  $\lfloor \phi/2 \rfloor$  columns. Bunch and Hopcroft observed that columns in  $R$  are not used until we eliminate all columns from  $L$ .<sup>6</sup> Hence all updates to columns in  $R$  resulting from elimination of columns in  $L$  can be done once using fast matrix multiplication in  $O(|R|^\omega)$  time. By applying this scheme recursively one obtains an  $O(\phi^\omega)$  time algorithm.

**Generalizations** Section 12 gives  $O(\phi^\omega)$  time algorithms for finding  $f$ -factors in non-bipartite multigraphs. There are several things that need to be done differently. As discussed above the criteria for finding allowed edges – Corollary 9 – does not work any more. We need to work with the weaker notion of *removable edges*, i.e., the edges that can be removed from the graph so that it still contains an  $f$ -factor. This forces us to use a different approach, based on Harvey [17], which works with removable edges. This poses a new challenge, to handle multiple copies of edges in multigraphs, as removing such edges one by one could require  $O(\phi^3)$  time. To overcome this problem we use binary search with the Sherman-Morrison-Woodbury formula to remove multiple copies of an edge in one shot.

### 4.3 Weighted $f$ -factors

In this section we discuss how to find a maximum  $f$ -factor in a weighted bipartite graph. For the sake of simplicity we assume in this section that the weight function is non-negative, i.e.,  $w : E \rightarrow [0..W]$ . If this is not the case we can redefine  $w(ij) := w(ij) + W$ , what changes the weight of each  $f$ -factor by exactly  $Wf(V)/2 = W\phi$ . Let us start by recalling the dual problem for maximum  $f$ -factors. In this problem each vertex  $v$  is assigned a real-valued weight  $y(v)$ . We say that the dual  $y$  *dominates* the edge  $uv \in E$  when  $y(u) + y(v) \geq w(e)$ , or it *underrates* the edge  $uv \in E$  when  $y(u) + y(v) \leq w(uv)$ . The objective that we need to minimize in dual problem is

$$y(V, E) = \sum_{v \in V} f(v)y(v) + \sum_{uv \in E \text{ is underrated}} w(uv) - y(u) - y(v).$$

The dual  $y$  minimizes  $y(V, E)$ , when there exists an  $f$ -factor  $F$  such that  $F$  contains only underrated edges, whereas its complement contains only dominated edges. Observe that when we are given the minimum dual  $y$ , then the above  $f$ -factor  $F$  is a maximum weight  $f$ -factor. On the other hand, in

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<sup>6</sup>In their paper the elimination proceeds row by row, whereas it is nowadays more usual to present Gaussian elimination on columns.

order to construct such maximum  $f$ -factor we need to take into it every strictly underrated edge and arbitrary *tight* edges, i.e., edges  $uv \in E$  for which  $y(u) + y(v) = w(uv)$ . Hence, we can observe the following.

**Lemma 12.** *Given an optimal dual function  $y$ , a maximum  $f$ -factor of a bipartite multigraph can be constructed in  $O(\phi^\omega)$  time.*

*Proof.* Let  $U$  be equal to the set of underrated edges with respect to  $y$ . Set  $f'(v) = f(v) - d(v, U)$ . Using Algorithm 1 find an  $f'$ -factor  $T$  over the set of tight edges with respect to  $y$ . The maximum  $f$ -factor is equal to multiset sum of  $U$  and  $T$ , i.e., to  $U \uplus T$ . ■

This lemma shows that given an algorithm for finding unweighted  $f$ -factors all we need to know is an optimal dual. Such an optimal dual can be obtained from the combinatorial interpretation as given in [14]. Let us define  $G^+$  to be  $G$  with additional vertex  $s \in V_1$  and new 0 weight edges  $su$ , for all  $u \in V_0$ . In  $G^+$  we set  $f(s) = 1$ . Let  $f_v$  be the degree constraint function defined to be identical to  $f$  except for  $f_v(v) = f(v) + (-1)^i$ , where  $v \in V_i$ . Let  $F_v$  be a maximum  $f_v$ -factor in  $G^+$ . To show that  $F_v$  always exists take  $F$  to be any  $f$ -factor in  $G$ . Now, when  $v \in V_0$  then  $F + sv$  is an  $f_v$ -factor, whereas when  $v \in V_1 - s$  then for any  $uv \in F$ ,  $F - uv + su$  is an  $f_v$ -factor

**Theorem 13** ([14]). *For a bipartite multigraph with an  $f$ -factor, optimal duals are given by  $y(v) = (-1)^i w(F_v)$  for  $v \in V_i$ .*

Hence, in order to construct optimal dual we need to know weights  $w(F_v)$ , for all  $v \in V_0 \cup V_1$ . At first sight it might seem that we did not gain anything, as instead of finding one  $F$  factor now we need to find all factors  $F_v$ . However, we do not need to find these factors. We only need to know their weights, which is much easier. And the following lemma shows that we just need to know  $w(F_v)$  for one side of the bipartite graph.

**Lemma 14.** *For a bipartite multigraph with an  $f$ -factor, let  $y(v)$  be an optimal dual for each  $v \in V_1$ . An optimal dual  $y(u)$  for  $u \in V_0$  is equal to the largest value  $y_u$  that makes at least  $f(u)$  edges incident to  $u$  underrated, i.e.,  $|\{uv \in E : y_u \leq w(uv) - y(v)\}| \geq f(u)$ .*

*Proof.* Observe that there are at least  $f(u)$  underrated edges incident to  $u$  with respect to optimal dual  $y$ , as each maximum  $f$ -factor needs to contain only underrated edges. On the other hand, the complement contains at least  $d(u) - f(u)$  dominated edges. This fixes the largest possible value for  $y(u)$  as the value given in the lemma. ■

Now consider a simple bipartite graph  $G$  and similarly to (1) define  $B(G)$  as

$$B(G)_{i,r,j,c} = \begin{cases} z^{w(ij)} x_r^{ij} y_c^{ij} & ij \in E, \\ 0 & \text{otherwise,} \end{cases}$$

where  $z$  is a new indeterminate. Theorem 5 shows that there is a mapping  $\mathcal{F}$  from terms of  $\det(B(G))$  onto  $f$ -factors in  $G$ . Consider a term  $\sigma$  in  $\det(B(G))$ . Observe that its degree in  $z$  is equal to the weight of  $\mathcal{F}(\sigma)$  because the powers of  $z$  get added in the multiplication. For a polynomial  $p$ , denote the degree of  $p$  in  $z$  by  $\deg_z(p)$ . We obtain the following observation.

**Corollary 15.** *For a simple bipartite graph  $G$ ,  $\deg_z(\det(B(G)))$  equals the weight of a maximum  $f$ -factor in  $G$ .*

To compute  $f_v$ -factors in  $G^+$  we use the following auxiliary graph. Let  $G_*$  be  $G^+$  with an additional vertex  $t \in V_0$  that is joined to every vertex of  $V_1 - s$  by an edge of weight zero. Set  $f(t) = 1$ .

**Lemma 16.** *Every  $v \in V_1$  satisfies  $\deg_z(\text{adj}(B(G_*))_{v,0,t,0}) = w(F_v)$ .*

*Proof.* Observe that

$$\text{adj}(B(G_*))_{v,0,t,0} = (-1)^{n(t,0)+n(v,0)} \det(B(G_*)^{t,0,v,0}),$$

where  $B(G_*)^{t,0,v,0}$  is the matrix  $B(G_*)$  with row  $t, 0$  and column  $v, 0$  removed and  $n(i, r)$  gives the order of rows and columns indexed by pairs  $i, r$ . By Theorem 5 we know that  $\det(B(G_*))$  consists of terms corresponding to  $f$ -factors in  $G_*$ . Hence the above equality shows terms of  $\text{adj}(B(G_*))_{v,0,t,0}$  correspond to  $f$ -factors that use edge  $tv$ , but with this edge removed. These are exactly the  $f_v$ -factors in  $G^+$ , because forcing the  $f$ -factor to use edge  $tv$  effectively decreases  $f(v)$  by 1. As we observed in Corollary 15 the degree of  $z$  equals the total weight of corresponding  $f$ -factor. The lemma follows. ■

Recall that the adjoint of a nonsingular matrix  $A$  is  $\det(A)A^{-1}$ . The lemma shows we are interested in column  $t, 0$  of the adjoint. So let  $e_{t,0}$  be a unit vector, with 1 in row  $t, 0$  and zeroes elsewhere. Then the desired weights are found in the vector  $\text{adj}(B(G_*))e_{t,0} = \det(B(G_*))B(G_*)^{-1}e_{t,0}$ . This leads to the following algorithm to find optimal duals for weighted bipartite  $f$ -factors.

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**Algorithm 2** An  $\tilde{O}(W\phi^\omega)$  time algorithm to find optimal duals  $y$  in a bipartite graph  $G, f$ .

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- 1: Let  $B(G_*)$  be  $\phi \times \phi$  matrix representing  $G_*$
  - 2: Replace  $x$  and  $y$  variables in  $B(G_*)$  by random values from  $\mathcal{Z}_p$  for prime  $p = \Theta(\phi^3)$  to obtain  $B$
  - 3: Compute vector  $a := \text{adj}(B)e_{t,0} = \det(B)B^{-1}e_{t,0}$ .   ▷ requires  $\tilde{O}(W\phi^\omega)$  time using Theorem 3
  - 4: **for**  $v \in V_1$  **do**
  - 5:    $(\deg_z(a_v) = \deg_z(\text{adj}(B(G_*))_{v,0,t,0}))$  with probability  $\geq 1 - \frac{1}{\phi^2}$    ▷ by Cor. 1
  - 6:   Set  $w(F_v) := \deg_z(a_v)$    ▷ equality holds by Lemma 16
  - 7:   Set  $y(v) := w(F_v)$    ▷  $y(v)$  is optimal by Theorem 13
  - 8: **end for**   ▷ by union bound all  $y(v)$  are correct with probability  $\geq 1 - \frac{1}{\phi}$
  - 9: **for**  $u \in V_0$  **do**
  - 10:   Set  $y(u) := \max\{y_u : |\{uv \in E : y_u \leq w(uv) - y(v)\}| \geq f(u)\}$
  - 11:   ▷  $y(u)$  is optimal by Lemma 14
  - 12: **end for**
- 

Combining the above algorithm with Lemma 12 we obtain an  $\tilde{O}(Wn^\omega)$  time algorithm for maximum  $f$ -factors in weighted bipartite graphs. It can be observed that this development works for bipartite multi-graphs as well, when one changes the definition of  $B(G)$  to

$$B(G)_{i,r,j,c} = \sum_{k=1}^{\mu(ij)} z^{w(ij,k)} x_r^{ij,k} y_c^{ij,k}.$$

As in Section 4, some problem instances require a more careful construction of  $B$ . But the algorithm of Section 4 is easily extended. First note that  $b$ -matching problems can be handled using the

algorithm of Section 4 unchanged, since every copy of a fixed edge  $ij$  has the same weight  $w(ij, k) = w(ij)$ . The most general case ( $f$ -factor problems with high multiplicities and parallel edges of different weights) is easily handled as follows. Decompose  $G$  into multigraphs  $G_w$  containing the edges of weight  $w$ , for  $w = 0, \dots, W$ . So  $B(G) = \sum_{w=0}^W z^w B(G_w)$ , and analogously,  $B = \sum_{w=0}^W z^w B_w$  (where all  $B_w$ 's use the same random values for the  $x$  and  $y$  variables). Compute each  $B_w$  using the algorithm of Section 4 and combine. Each  $B_w$  is found in time  $O(\phi^\omega)$  so the total time is  $O(W\phi^\omega)$ .

**Generalizations** The relation between the primal and dual problems in the bipartite case is considerably simpler than the general case. The latter not only contains dual variables for vertices, but also for subsets of vertices. Such subsets with non-zero dual value are called blossoms. We can prove these blossoms are nested and so form a blossom tree. Moreover, for each blossom we need to find a set of spanned edges and incident edges that are all underrated. This cannot be done in such a simple way as Lemma 14. But again knowing the weights of maximum  $f_v$ -factors turns out to be enough. The procedure that deduces all this information is highly nontrivial and is described in Section 8. This section is preceded with two special, simpler cases. First Section 5 considers maximum  $b$ -matchings, where there are no underrated incident edges. Second Section 6 considers shortest paths in undirected graphs with negative weights, where each blossom has exactly one underrated incident edge. Finally Section 13 generalizes Algorithm 2 to compute  $w(F_v)$  for non-bipartite graphs. Alternatively for the special case of shortest paths  $w(F_v)$  can be computed combinatorially as in Section 10.

#### 4.4 Min-cost max-flow

This section presents network flow algorithms. We discuss maximum flows and minimum cost maximum flows, both in vertex-capacitated networks.

We are given a directed network  $N = (V, E)$ , with source  $s$  and sink  $t$ ,  $s, t \in V$ . For convenience let  $V^-$  denote the set of nonterminals,  $V - \{s, t\}$ . The edges and nonterminal vertices have integral capacities given by  $c : V^- \cup E \rightarrow [1..D]$ . Let  $g : V \times V \rightarrow \mathbb{Z}$  be a flow function. Besides the standard *edge capacity* and *flow conservation* constraints we have *vertex capacity* constraints, i.e., for each vertex  $v \neq s, t$  we require

$$\sum_{u \in V} g(u, v) \leq c(v).$$

We begin by constructing a bipartite graph  $G_N$  whose maximum  $f$ -factor has weight equal to the value of a maximum flow in  $N$ . Wlog assume that no edge enters  $s$  or leaves  $t$ . The construction proceeds as follows:

- for each  $v \in V^-$  place vertices  $v_{in}, v_{out}$  in  $G_N$ ;
- also place vertices  $s_{out}, t_{in}$  in  $G_N$ ;
- for each  $v \in V^-$  add  $c(v)$  copies of edge  $v_{in}v_{out}$  to  $G_N$ ;
- for each  $(u, v) \in E$  add  $c(u, v)$  copies of edge  $u_{out}v_{in}$  to  $G_N$ ;
- for  $v \in V^-$ , set  $f(v_{in}) = f(v_{out}) = c(v)$ ;
- set  $f(s_{out}) = f(t_{in}) = c(V^-)$ ;
- set  $w(e) = 1$  for each edge  $e$  leaving  $s_{out}$  and  $w(e) = 0$  for every other edge of  $G_N$ ;
- add  $c(V^-)$  copies of edge  $s_{out}t_{in}$  to  $G_N$ , all of weight 0.

Note that in addition to the weight 0 edges  $s_{out}t_{in}$ ,  $G_N$  may contain edges  $s_{out}t_{in}$  of weight 1 corresponding to an edge  $st \in E$ .

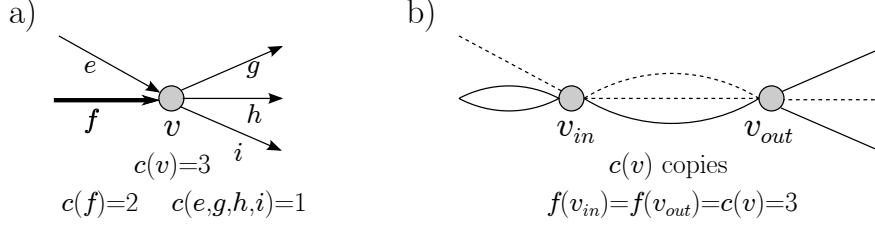


Figure 1: A vertex  $v$  of capacity  $c(v) = 3$  in  $N$  is represented in  $G_N$  by two vertices  $v_{in}$  and  $v_{out}$  connected by 3 edges. The  $f$ -factor in (b) is marked with solid edges. Observe that the  $f$ -factor must choose the same number of edges going in and out of  $v$ .

**Corollary 17** ([15]). *Let  $N$  be the flow network. The weight of the maximum  $f$ -factor in  $G_N$  is equal to the maximum flow value in  $N$ .*

*Proof.* The main idea of the reduction is shown in Figure 1. Observe that  $f$ -factors in  $G_N$  correspond to integral flows in  $N$  that fulfill the flow conservation constraints. Moreover the edge capacities are not exceeded since an edge cannot be used by an  $f$ -factor more times than its capacity. Similarly a vertex cannot be used more times than its capacity. The only edges with non-zero weights are edges incident to  $s_{out}$ , so the maximum  $f$ -factor maximizes the amount of flow leaving  $s$ . Finally all this flow must wind up at  $t$ , since any  $v \neq s, t$  has an equal number of  $f$ -edges incident to  $v_{in}$  and  $v_{out}$ . ■

Observe that  $G_N$  has  $f(V) \leq 4c(V^-)$ . So the algorithm of Section 4.3 uses  $\tilde{O}(c(V^-)^\omega) = \tilde{O}((Dn)^\omega)$  time to find a maximum flow in a network.

Now assume that an edge  $(u, v)$  of  $N$  has a cost  $a_{u,v} \in [-W..W]$ , i.e., the cost of sending  $t$  units of flow on edge  $(u, v)$  is linear and equals  $ta_{u,v}$ . First find the maximum flow value  $f_{max}$  in  $N$ . Then modify the construction of  $G_N$  to  $G_{N,a}$  in the following way:

- add vertices  $s_{in}$  and  $t_{out}$ ;
- add  $c(V^-)$  copies of edges  $s_{in}s_{out}$  and  $t_{in}t_{out}$ ;
- set  $f(s_{in}) = f(t_{out}) = c(V^-)$  and  $f(s_{out}) = f(t_{in}) = c(V^-) + f_{max}$ ;
- for each copy of the edge  $u_{out}v_{in}$  set  $w(u_{out}v_{in}) = -a_{u,v}$ .

In Corollary 17 we observed that  $f$ -factors in  $G_N$  correspond to feasible flows in  $N$ . The new vertices  $s_{in}$  and  $t_{out}$  allow an  $f$ -factor to model flow along cycles containing  $s$  and  $t$ . The new values of  $f$  at the terminals make the corresponding flow from  $s$  to  $t$  have value  $f_{max}$ . Thus  $f$ -factors in  $G_{N,a}$  correspond to maximum flows in  $N$ , and since costs of edges are negated between both networks, we have the following.

**Corollary 18** ([15]). *Let  $N$  be a flow network with linear edge costs. A maximum  $f$ -factor in  $G_{N,a}$  has weight equal to the minimum cost of a maximum flow in  $N$ .*

Now the algorithm of Section 4.3 gives an  $\tilde{O}(Wc(V^-)^\omega) = \tilde{O}(W(Dn)^\omega)$  time algorithm to find a min-cost max-flow.

Let us extend this reduction to convex cost functions. Assume the cost of sending  $t$  units of flow on an edge  $(u, v)$  is given by a convex function  $a_{u,v}(t)$  such that marginal costs satisfy  $m_{u,v}(t) = a_{u,v}(t) - a_{u,v}(t - 1) \in [-W..W]$ . As usual for this scenario [1] we assume  $a_{u,v}$  is linear between successive integers. This ensures that there exists an integral optimal solution. We encode

such cost functions in the graph by assigning different costs to each copy of an edge, i.e., the  $k$ th copy of edge  $u_{out}v_{in}$  has cost  $w(u_{out}v_{in}, k) = -m_{u,v}(k)$  for  $1 \leq k \leq c(u, v)$ .

**Lemma 19.** *Let  $N$  be a flow network with convex edge costs. A maximum  $f$ -factor in  $G_{N,a}$  has weight equal to the minimum cost of a maximum flow in  $N$ .*

*Proof.* The maximum  $f$ -factor in  $G_{N,a}$ , when using  $t$  copies of edges between  $u_{out}$  and  $v_{in}$ , will use the most expensive copies. The convexity of  $a_{u,v}$  implies  $m_{u,v}(t)$  is a non-decreasing function. Hence we can assume that edges with costs  $-m_{u,v}(1), \dots, -m_{u,v}(t)$  are used. These costs sum to  $-a_{u,v}(t)$ . ■

Clearly this extension does not change the running time of our algorithm. So we find a min-cost max-flow for convex edge costs in time  $\tilde{O}(W(Dn)^\omega)$ .

It is also easy to incorporate lower bounds on flow into the reduction. Suppose that in addition to the upper bound function  $c$  we have a lower bound function  $\ell : V^- \cup E \rightarrow [0..D]$ ;  $\ell$  restricts the flow on edges and flow through nonterminal vertices in the obvious way. To model  $\ell$ , change the multiplicity of  $v_{in}v_{out}$  to  $c(v) - \ell(v)$  and the multiplicity of  $u_{out}v_{in}$  to  $c(u, v) - \ell(u, v)$ ; in addition for  $v \in V^-$ ,  $f(v_{in})$  becomes  $c(v) - \sum_u \ell(u, v)$  and  $f(v_{out})$  becomes  $c(v) - \sum_u \ell(v, u)$ . It is easy to see that the correspondence between  $f$ -factors and feasible flows is maintained, as is the correspondence between weights and costs. (In particular starting from an  $f$ -factor, adding  $\ell(u, v)$  copies of each edge  $u_{out}v_{in}$  gives a subgraph that obviously corresponds to a flow satisfying all lower bounds on edges. It is easy to check this flow also satisfies all lower bounds on vertices.)

The construction can also be generalized to bidirected flows. In directed graphs an undirected edge can have two orientations, in-out and out-in, whereas a bidirected graph allows four possible orientations in-in, out-in, in-out and out-out. (We also allow loops, especially of type in-in and out-out.) A bidirected network  $G$  gives a non-bipartite graph  $G_N$ . However as we show in the remainder of this paper, the  $f$ -factor problem for non-bipartite graphs can be solved in the same time bounds as the bipartite case.

## 5 Weighted $b$ -matching

This section gives our algorithm to find a maximum  $b$ -matching.  $b : V \rightarrow \mathbb{Z}_+$  can be an arbitrary function. But we remark that our algorithm is of interest even for the case  $b \equiv 1$  (ordinary matching): It achieves the same time bound as [7], and is arguably simpler in both derivation and algorithmic details.<sup>7</sup>

We start with two remarks that modify the definition of the problem. First it is most often convenient to use the language of multigraphs. In this view we think of  $G$  as a multigraph with an unlimited number of copies of each edge. Specifically each  $e \in E$  has  $1 + \max_v b(v)$  copies, each with the same weight  $w(e)$ . (Also recall from Section 2 that  $G$  may have loops.) A  $b$ -matching is a subgraph of this multigraph.

Second, our algorithm actually works on critical graphs, defined as follows. Given a function  $b : V \rightarrow \mathbb{Z}^+$ , for each  $v \in V$  define  $b_v : V \rightarrow \mathbb{Z}_+$  by decreasing  $b(v)$  by 1 (keep all other values unchanged). A graph is  $b$ -critical if it has a perfect  $b_v$ -matching for every  $v \in V$ . Given a  $b$ -critical

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<sup>7</sup>The reader may enjoy working through the derivation for the matching case  $b \equiv 1$ . There will be no need to work with a multigraph. This endeavor will show how the two problems differ, especially in some details introduced by multigraphs.

graph, our algorithm produces a "blossom tree" from which, for any  $v \in V$ , a maximum  $b_v$ -matching can be easily extracted.

Such an algorithm can find a maximum  $b$ -matching as follows. Suppose we seek a maximum  $b$ -matching on  $G$ . Assume every  $b(v)$  is positive (discard 0-valued vertices). Form  $G'$  by adding a vertex  $s$  with  $b(s) = 1$ , plus edges  $sv$ ,  $v \in V(G)$  and loop  $vv$ , all of weight 0. Any  $v \in V + s$  has a  $b_v$ -matching. (For  $v = s$  take  $F$ , an arbitrary  $b$ -matching on  $G$ . For  $v \neq s$ , for any edge  $vw \in F$  take  $F - vw + ws$ . This is a  $b_v$ -matching even if  $vw$  is a loop.) So  $G'$  is  $b$ -critical and a maximum  $b_s$ -matching on  $G'$  is the desired  $b$ -matching.

## 5.1 The heaviest blossom

This section defines the  $\zeta$ -value of an edge and shows how it reveals the heaviest blossom (the set  $B$  defined after Lemma 22). Consider a weighted  $b$ -critical graph  $G$ . Recall that  $G$  may contain self-loops  $vv$ . All of what follows is valid even when there are such loops. Assume that each multiset of  $\leq \sum_v b(v)$  edges has a distinct weight, if we do not distinguish between parallel copies of the same edge. We can enforce this assumption by taking a very small  $\epsilon > 0$  and adding  $\epsilon^i$  to the weight of every copy of the  $i$ th edge of  $E$ . (Section 5.3 returns to the original unperturbed weights.)

Any vertex  $v$  has a maximum  $b_v$ -matching  $F_v$ .  $F_v$  is unique up to parallel copies of the same edge, by the perturbed weight function. Wlog assume further that any two matchings  $F_u$  and  $F_v$  have as many common edges as possible (i.e., for any  $xy \in E$  they use as many of the same copies of  $xy$  as possible).

We start with a well-known principle. Take two vertices  $u, v$ . Call a trail *alternating* (for  $u$  and  $v$ ) if as we traverse it the edges alternate between  $F_u - F_v$  and  $F_v - F_u$ .

**Lemma 20.** *For any two vertices  $u, v$ ,  $F_u \oplus F_v$  is an alternating  $uv$ -trail that starts with an edge of  $F_v - F_u$  and ends with an edge of  $F_u - F_v$ .*

*Proof.* First we show that  $F_u \oplus F_v$  contains an alternating trail as described in the lemma. (Then we show that trail constitutes all of  $F_u \oplus F_v$ .) Let  $T$  be a maximal length alternating trail that starts at  $u$  with an edge of  $F_v - F_u$ . Such an edge exists since  $u$  has greater degree in  $F_v$  than  $F_u$ . Let  $T$  end at vertex  $x$ .

If  $x \neq u, v$  then, since  $d(x, F_u) = d(x, F_v)$ , we can extend  $T$  with an unused alternating edge. (This is true even if  $T$  has a previous occurrence of  $x$ .) If  $x = u$  then we can extend  $T$  with an unused alternating edge – the argument is the same using the facts that  $d(u, F_v) = d(u, F_u) + 1$  and the first edge of  $T$  is in  $F_v - F_u$ . Suppose  $x = v$ . If the last edge of  $T$  is in  $F_v - F_u$  we can extend  $T$  with an unused edge of  $F_u - F_v$  since  $d(v, F_u) = d(v, F_v) + 1$ . The remaining possibility is that  $T$  ends at  $v$  with an edge of  $F_u - F_v$ , thus giving the desired trail.

Now we show there are no other edges.  $F_v$  is the disjoint union of its edges in  $T$  and a multiset of edges  $R$ . ( $R$  may contain copies of edges in  $T$ .) We claim any vertex  $x$  satisfies

$$d(x, R) = d(x, F_v) - d(x, T \cap F_v) \tag{3}$$

$$= b(x) - \lceil d(x, T)/2 \rceil. \tag{4}$$

Observe that this claim completes the proof of the lemma: Define the function  $b'$  by setting  $b'(x)$  to the quantity of (4). Thus  $R$  is a  $b'$ -matching, in fact a maximum  $b'$ -matching. By symmetry  $F_u$  is the disjoint union its edges in  $T$  and a maximum  $b'$ -matching. The perturbed weight function implies that  $b'$ -matching is also  $R$ . Thus  $F_u$  and  $F_v$  agree outside of  $T$ , so  $F_u \oplus F_v = T$ .

To prove the claim, (3) holds by definition. For (4) first observe that  $x \neq v$  implies

$$d(x, F_v) = b(x) \text{ and } d(x, T \cap F_v) = \lceil d(x, T)/2 \rceil.$$

In proof the first relation follows from definition of  $F_v$ . For  $x \neq u, v$  the second relation holds because the edges through  $x$  alternate. For  $x = u$  the second relation holds because the first edge of  $T$  is in  $F_v$ , and all other pairs of edges through  $u$  alternate. Substituting the two relations into (3) gives (4).

If  $x = v$  then  $d(v, F_v) = b(v) - 1$  and  $d(v, T \cap F_v) = (d(v, T) - 1)/2$ . The latter holds because the last edge of  $T$  is not in  $F_v$ . Furthermore all other pairs through  $v$  alternate. Again substituting these two relations into (3) gives (4). ■

In an arbitrary multigraph let  $C$  be an odd circuit containing a vertex  $u$ . Choose a traversal of  $C$  that starts at  $u$ . Define  $C_u$  to consist of alternate edges in this traversal, omitting the first edge at  $u$  as well as the last. When  $C$  contains  $> 2$  edges at  $u$ ,  $C_u$  will not be unique. But suppose the traversal starts with edge  $uv$ , and we define  $C_v$  by the same traversal only starting at  $v$ . Then  $C_u$ ,  $C_v$  and  $uv$  form a partition of  $C$  (since  $C_u$  starts by containing the edges of  $C - C_v$ , and this pattern continues until the traversal reaches  $v$ ). Also given any choice of a  $C_x$ , define  $C^x$  to consist of alternate edges of  $C$ , beginning and ending with the edge incident to  $x$ . Clearly  $C_x$  and  $C^x$  form a partition of the edges of  $C$ .

To define the central concept, for any edge  $uv$  let

$$\zeta(uv) = w(F_u) + w(F_v) + w(uv).$$

We shall see that  $\zeta$  gives the values of the optimum blossom duals as well as the structure of the optimum blossoms.

**Lemma 21.** *Any edge  $e$  of a  $b$ -critical graph belongs to an odd circuit of edges with  $\zeta$ -value  $\geq \zeta(e)$ .*

*Proof.* Let  $e = uv$ . Assume  $u \neq v$  else the lemma is trivial. Furthermore assume  $e \notin F_u \cup F_v$ . This is the crucial assumption! It is justified since there are  $b(v) + 1$  copies of  $e$ ; furthermore proving the lemma for this copy of  $uv$  proves it for every copy.

Let  $T$  be the trail of Lemma 20. It clearly has even length. Extend it by adding a copy of edge  $e$ . ( $T$  may already contain a different copy of  $e$ , one in  $F_u \oplus F_v$ .) We get an odd circuit  $C$ . Traverse  $C$  by starting with the first edge of  $T$  and ending with the edge  $uv \in C - T$ . We get  $C$  partitioned into  $C_u$ ,  $C_v$  and  $uv$ . Furthermore  $C_u = C \cap F_u$ ,  $C_v = C \cap F_v$ , and a set of edges  $R$  satisfies  $R = F_u - C = F_v - C$ . Thus

$$\zeta(uv) = w(F_u) + w(F_v) + w(uv) = w(C) + 2w(R).$$

Now take any edge  $rs \in C$ . Traverse  $C$  by starting with edge  $rs$ . For  $t \in \{r, s\}$  let  $H_t$  be the multiset  $R \cup C_t$ . It is easy to see  $H_t$  is a  $b_t$ -matching by comparing it with  $F_v = R \cup C_v$ . Then

$$\zeta(rs) \geq w(H_r) + w(H_s) + w(rs) = w(C) + 2w(R).$$

The two displayed equations show  $C$  is the desired circuit. (In this argument some vertices  $t$  may have  $C_t$  multiply defined. That's OK.) ■

Let  $\zeta^*$  be the maximum value of  $\zeta$ . Let  $E^*$  be the set of edges of  $\zeta$ -value  $\zeta^*$ . We shall see that  $E^*$  is essentially the heaviest blossom and  $\zeta^*$  its dual value.

**Lemma 22.** *Any edge  $e = vw \in E^*$  belongs to an odd circuit  $C \subseteq E^*$ . Furthermore (i)  $F_v - C = F_w - C$  and (ii)  $F_v \cap (\gamma(v) \cup \delta(v)) \subseteq E^*$ .*

*Proof.* Lemma 21 shows the odd circuit  $C$  exists. To prove (i) recall from the proof that

$$w(C) + 2w(R) = \zeta^*$$

and  $F_v = R \cup C_v$ ,  $F_w = R \cup C_w$ . The last two equations imply (i).

For (ii) take any edge  $uv \in F_v$ . Keep  $C$  and  $R$  as already defined for  $vw$ . We can assume  $uv \in R$ , since otherwise  $uv \in C_v \subseteq C \subseteq E^*$ . Let  $H_u = R - uv + C^v$ . It is easy to see  $H_u$  is a  $b_u$ -matching by comparing it with  $F_v = R \cup C_v$  (note  $u$  may or may not belong to  $C$ ). Then

$$\zeta(uv) \geq w(H_u) + w(F_v) + w(uv) = w(C) + 2w(R) = \zeta^*.$$

This implies equality holds and proves (ii). ■

Call a connected component  $B$  of  $E^*$  *nontrivial* if it spans at least one  $\zeta^*$ -edge.<sup>8</sup> ( $B$  may consist of a single vertex  $v$  with one or more loops  $vv$ .) The next lemma shows that  $B$  behaves like a blossom, i.e., it can be shrunk to a single vertex. We begin the proof with two observations. First, any two vertices  $v, w \in B$  have

$$F_v - E^*(B) = F_w - E^*(B). \quad (5)$$

In proof, since  $B$  is connected we need only show (5) when  $vw \in E^*(B)$ . That case follows by applying Lemma 22(i) with  $C \subseteq E^*(B)$ .

Next observe for any  $v \in B$ ,

$$F_v \cap (\gamma(B) \cup \delta(B)) \subseteq E^*(B). \quad (6)$$

This follows since an edge  $xy$  in the left set but not in the right, with  $x \in B$ , belongs to  $F_v - E^*(B) = F_x - E^*(B)$  by (5). But Lemma 22(ii) shows  $xy \in F_x \cap \delta(x) \subseteq E^*$ . This implies  $y \in B$  and  $xy \in E^*(B)$ , contradiction.

**Lemma 23.** *For any vertex  $v$ ,  $|F_v \cap \delta(B)|$  equals 0 for  $v \in B$  and 1 for  $v \notin B$ .*

*Proof.* For  $v \in B$  this follows from (6). So suppose  $v \notin B$ . Choose any vertex  $t \in B$ . Lemma 20 shows  $F_v \oplus F_t$  is an alternating  $vt$ -trail. Let  $T$  be the subtrail from  $v$  to the first vertex of  $B$ , say vertex  $x$ , and let  $e$  be the last edge of  $T$ .  $e \in F_v$  ( $e \notin F_t$  by the lemma for  $t$ , i.e.,  $F_t \cap \delta(B) = \emptyset$ ). Applying (5) to  $t$  and  $x$  shows  $F_v \oplus F_x$  contains  $T$ . So Lemma 20 applied to  $v$  and  $x$  shows  $F_v \oplus F_x = T$ . (More precisely, the first part of the proof of Lemma 1 could have chosen the current  $T$  as its  $vx$  trail. The second part of the proof shows  $F_v \oplus F_x$  has no other edges.) Thus  $e$  is the unique edge of  $F_v$  incident to  $B$ . ■

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<sup>8</sup>For ordinary matching  $B$  is a cycle that comprises all of  $E^*$ .

## 5.2 Iterating the construction

The last lemma generalizes [14, Lemma 3.2] for ordinary matching. The rest of the derivation closely parallels ordinary matching, as described in [14] and [7]. In a nutshell, the optimum blossoms and their duals are found by running the shrinking procedure of [14]. The nesting of the contracted blossoms gives a "blossom tree"  $\mathcal{B}$ . Any desired maximum  $b_v$ -matching ( $v \in V(G)$ ) can be constructed by a top-down traversal of  $\mathcal{B}$  that finds a perfect  $b'$ -matching (for appropriate  $b'$ ) at each node of  $\mathcal{B}$ . This last step is the biggest difference from ordinary matching: In ordinary matching the edges found at each node of  $\mathcal{B}$  are simply alternate edges of a cycle. For completeness we present all of these remaining details, of course modified for  $b$ -matching.

Consider a nontrivial connected component  $B$  of  $E^*$ , as above. Contract  $B$  to a vertex  $\overline{B}$ , with  $\overline{G}$  the resulting multigraph. Assume a contraction operation can create parallel edges but not loops. (Thus the graph changes even when  $B$  is a single vertex  $v$  with  $E^*$  consisting of loops  $vv$ .) Extend the degree-constant function  $b$  to  $\overline{G}$  by setting  $b(\overline{B}) = 1$ . Lemma 23 shows  $\overline{G}$  is  $b$ -critical. From now on  $V$  designates the vertex set of the original given graph.

Next we define a weight function on  $\overline{G}$ . For convenience we designate edges of  $\overline{G}$  by their corresponding edge in  $G$  (i.e., an edge of  $\overline{G}$  is written as  $uv$  where  $u, v \in V$  and possibly one of them belongs to  $B$ ). For  $v \in B$  let

$$B_v = F_v \cap \gamma(B).$$

For edge  $uv$  in  $\overline{G}$  define a weight  $\overline{w}(uv)$  by

$$\begin{aligned} \overline{w}(uv) &= w(uv) & u, v \notin B, \\ &= w(uv) + w(B_v) & v \in B. \end{aligned}$$

This definition preserves the structure of  $G$  in the following sense. For any vertex  $v$  let  $\overline{v}$  be its image in  $\overline{G}$ . Let  $F_{\overline{v}}$  denote the (unique) maximum  $b_{\overline{v}}$ -matching in  $\overline{G}$ . For a fixed vertex  $v$ , let vertex  $x \in B$  be  $v$  if  $v \in B$ , else the end of an edge of  $F_v$ .  $x$  is uniquely defined by Lemma 23.

**Lemma 24.** (i)  $F_v = F_{\overline{v}} \cup B_x$ .

(ii) Any edge of  $\overline{G}$  has the same  $\zeta$ -value in  $G$  and  $\overline{G}$ .

*Proof.* (i) Let  $H_{\overline{v}}$  be the image of  $F_v$  in  $\overline{G}$ . Lemma 23 shows  $H_{\overline{v}}$  is a  $b_{\overline{v}}$ -matching. Furthermore  $F_v = H_{\overline{v}} \cup B_x$  (for  $v \notin B$  this follows from the optimality of  $F_v$ ). So we must show  $H_{\overline{v}} = F_{\overline{v}}$ . Clearly it suffices to show

$$\overline{w}(F_{\overline{v}}) \leq \overline{w}(H_{\overline{v}}). \quad (7)$$

Suppose  $v \in B$ .  $F_{\overline{v}} \cup B_v$  is a  $b_v$ -matching. Since it weighs no more than  $F_v$ ,  $w(F_{\overline{v}}) \leq w(H_{\overline{v}})$ . This is equivalent to (7) since neither set contains an edge incident to  $B$ .

Suppose  $v \notin B$ . The definition of  $\overline{w}$  shows  $w(F_v) = \overline{w}(H_{\overline{v}})$ , as well as  $\overline{w}(F_{\overline{v}}) \leq w(F_v)$  (by the optimality of  $F_v$ ). These relations combine to give (7).

(ii) Part (i) shows  $w(F_v)$  is  $\overline{w}(F_{\overline{v}})$  for  $v \notin B$  and  $\overline{w}(F_{\overline{v}}) - w(B_v)$  for  $v \in B$ . The two cases of (ii) (depending on whether or not the edge is in  $\delta(B)$ ) follow. ■

The following *shrinking procedure* [14] iterates the construction of  $\overline{G}$ . It also constructs a tree that represents the nesting of the contracted blossoms:

Start by creating a one-node tree for each vertex of  $V$ . Then repeat the following step until the graph consists of one vertex:

Let  $B$  be a nontrivial connected component of  $E^*$ . Form a tree whose root represents  $B$ ; the subtrees of the root are the trees that represent the vertices of  $B$  in the current graph. Then change the current graph by contracting  $B$  and defining  $b$  and  $\overline{w}$  as described above for  $\overline{G}$ .

The construction can be iterated since as noted each  $\overline{G}$  is  $b$ -critical. Lemma 24(ii) shows  $\zeta^*$  never increases from one iteration to the next.

Call the final tree the *blossom tree*  $\mathcal{B}$  of  $G$ , and each nonleaf a *blossom* of  $\mathcal{B}$ . A child  $C$  of  $B$  is either a *blossom-child* or a *vertex-child* (i.e., a leaf of  $\mathcal{B}$ ). Note that a vertex of  $V$  can occur in  $\mathcal{B}$  as a singleton blossom as well as a leaf. For any node  $B$  of  $\mathcal{B}$  let  $V(B)$  denote the set of leaf descendants of  $B$ .

The next goal is to describe how the edges of any  $F_v$ ,  $v \in V$ , are distributed in  $\mathcal{B}$ . We begin with several definitions. Consider the iteration in the shrinking procedure that ends by contracting  $B$ . Let  $G_B$  be the graph at the start of the iteration. Thus each child  $C$  of  $B$  corresponds to a vertex of  $B$  in the graph  $G_B$ . Let  $\zeta(B)$  be the value of  $\zeta^*$  in this iteration. Let  $E^*(B)$  be the corresponding set of edges (i.e., the edges of  $\zeta$ -value  $\zeta(B)$  that join two vertices of  $B$  in  $G_B$ ).

We now generalize Lemma 24(i) to any graph  $G_B$ . Consider any vertex  $v \in V(B)$ . As before let  $\overline{v}$  be the image of  $v$  in  $G_B$  (so  $\overline{v}$  is a child of  $B$ ).  $F_{\overline{v}}$  denotes the unique maximum  $b_{\overline{v}}$ -matching of  $G_B$ . For each child  $C$  of  $B$  let  $x \in V(C)$  be the vertex that is either  $v$  (if  $C = \overline{v}$ ) or is the end of an edge of  $F_{\overline{v}}$ .  $x$  is uniquely defined since a blossom  $C$  has  $b(C) = 1$ , and there is no choice for  $x$  if  $C$  is a vertex of  $V$ . We claim

$$F_v \cap \gamma(V(B)) = F_{\overline{v}} \cap \gamma(V(B)) \cup \bigcup_{C \text{ a child of } B} F_x \cap \gamma(V(C)). \quad (8)$$

The import of (8) is that it defines the entire set  $F_v$ . Specifically let  $\mathcal{V}$  be the root node of  $\mathcal{B}$ . Clearly any  $v \in V$  has  $F_v = F_v \cap \gamma(V(\mathcal{V}))$ . So the entire set  $F_v$  is defined by applying (8) to  $\mathcal{V}$  and then recursively to the children of  $\mathcal{V}$ .

Also note that in (8) if  $C$  is a vertex-child of  $B$  then the expression  $F_x \cap \gamma(V(C))$  is empty, by convention.

To prove (8) first observe  $F_v \cap E(G_B) = F_{\overline{v}}$ . This follows by repeated applications of Lemma 24(i), which shows the edges of  $F_v$  in  $\overline{G}$  form  $F_{\overline{v}}$ .

The observation justifies the first term  $F_{\overline{v}} \cap \gamma(V(B))$  in (8). To complete the proof note that the remaining edges of  $F_v$  are contained in the various blossom-children  $C$  of  $B$ . We show these edges are given by the terms  $F_x \cap \gamma(V(C))$  in (8). If  $C = \overline{v}$  this is obvious since  $x = v$ . If  $C \neq \overline{v}$  then  $F_{\overline{v}} \cap \delta(C)$  is a unique edge incident to  $x$ . The optimality of  $F_v$  implies it agrees with  $F_x$  in  $\gamma(V(C))$ . (8) follows.

(6) shows that in the graph  $G_B$ ,  $F_{\overline{v}} \cap \gamma(B) \subseteq E^*(B)$ . Applying this recursively with (8) shows  $F_v$  consists entirely of  $\zeta^*$ -edges, more precisely, in any blossom  $B$ , the edges of  $F_v$  that join children of  $B$  (or that are loops incident to a vertex that is a blossom-child of  $B$ ) are in  $\zeta(B)$ .

### 5.3 The efficient $b$ -matching algorithm

An efficient algorithm cannot work with the perturbed weight function. So suppose we execute the shrinking procedure, starting with the original unperturbed weight function  $w$ . Each iteration will find a connected component  $B$  that was found in the construction of  $\mathcal{B}$ . More precisely an iteration that contracts blossom  $C$  of  $\mathcal{B}$  will be skipped iff the parent of  $C$  in  $\mathcal{B}$ , say  $B$ , has  $\lfloor \zeta(C) \rfloor = \lfloor \zeta(B) \rfloor$ .

So the shrinking procedure will construct a tree  $\mathcal{W}$  that is a contraction of  $\mathcal{B}$  (more precisely an edge of  $\mathcal{B}$  from parent  $B$  to child  $C$  is contracted iff it satisfies the above relation  $\lfloor \zeta(C) \rfloor = \lfloor \zeta(B) \rfloor$ ).

For any blossom  $B$  of  $\mathcal{W}$ , define the graph  $G_B^*$  as follows. Consider the iteration of the shrinking procedure that contracts  $B$ . The vertices of  $G_B^*$  are the vertices of the current graph that are contained in  $B$ . (These vertices are the children of  $B$  in  $\mathcal{W}$ .) The edges of  $G_B^*$  are the edges  $E^*(B)$  in the shrinking procedure.

The following observation is key to our algorithm. It enables us to find the edges of any desired maximum  $b_v$ -matching that occur in each graph  $G_B^*$ . Take any vertex  $v \in V(B)$ . Let  $\bar{v}$  be the image of  $v$  in  $G_B^*$ . The edge set  $F_v \cap E^*(B)$  is a  $b_{\bar{v}}$ -matching of  $G_B^*$ . In proof first recall that  $F_v$  consists entirely of  $\zeta^*$ -edges. Then apply (8) to  $B$  and to each descendant of  $B$  (in  $\mathcal{B}$ ) that gets contracted into  $B$  when we form  $\mathcal{W}$  from  $\mathcal{B}$ .

We find a maximum  $b_v$ -matching using a recursive procedure  $b\_match$  that finds the desired edges in each graph  $G_B^*$ . For a blossom  $B$  of  $\mathcal{W}$  and a vertex  $v \in V(B)$ ,  $b\_match(B, v)$  finds a  $b_v$ -matching of  $\zeta^*$ -edges in the subgraph of  $G$  induced by vertices  $V(B)$ . It works as follows.

Let  $\bar{v}$  be the child of  $B$  that contains  $v$ . First find a  $b_{\bar{v}}$ -matching of  $G_B^*$ , say  $H_{\bar{v}}$ . ( $H_{\bar{v}}$  exists by the above key observation.) Add  $H_{\bar{v}}$  to the desired set. Then complete the desired set using recursive calls on the children of  $B$ . Specifically for each blossom-child  $C$  of  $B$  (in  $\mathcal{W}$ ), execute  $b\_match(C, x)$  where  $x \in V(C)$  is  $v$  (if  $C = \bar{v}$ ) or the end of an edge of  $H_{\bar{v}}$ .

As before let  $\mathcal{V}$  be the root of  $\mathcal{W}$ . Let  $H_v$  be the  $b_v$ -matching of the given graph  $G$  that is found by  $b\_match(\mathcal{V}, v)$ . In general  $H_v$  is not  $F_v$ . For instance in a recursive call  $b\_match(C, x)$ ,  $x$  may differ from the vertex  $x$  given by (8). But because  $H_v$  consists of  $\zeta^*$ -edges, we can prove  $H_v$  is optimum using duality, as follows.

As before for any blossom  $B$  of  $\mathcal{W}$  let  $\zeta(B)$  be the value  $\zeta^*$  when  $B$  is created. (Now  $\zeta(B)$  is an integer.) For any blossom  $B \neq \mathcal{V}$  let  $p(B)$  be its parent in  $\mathcal{W}$ . Let  $\mathcal{W}^-$  be the set of all blossoms of  $\mathcal{W}$ . Define functions  $y : V \rightarrow \mathbb{Z}$ ,  $z : \mathcal{W}^- \rightarrow \mathbb{Z}$  by

$$\begin{aligned} y(v) &= -w(F_v) & v \in V, \\ z(B) &= \begin{cases} \zeta(\mathcal{V}) & B = \mathcal{V}, \\ \zeta(B) - \zeta(p(B)) & B \in \mathcal{W}^- - \mathcal{V}. \end{cases} \end{aligned}$$

Recall our convention for summing functions (Section 2), e.g., an edge  $e = uv$  has  $y(e) = y(u) + y(v)$ . Any blossom  $B$  has  $\zeta(B) = z\{A : V(B) \subseteq V(A)\}$ . The definition of  $\zeta$  shows any edge  $e$  of  $G$  has  $w(e) = y(e) + \zeta(e)$ . So  $e \in E^*(B)$  implies

$$w(e) = y(e) + \zeta(B) = y(e) + z\{A : e \subseteq V(A)\}. \quad (9)$$

Let  $F$  be either  $F_v$  or  $H_v$ . Both sets  $F$  weigh

$$\begin{aligned} \sum_{e \in F} w(e) &= \sum_{e \in F} y(e) + z\{A : e \subseteq V(A)\} \\ &= (b_v y)(V) + \sum_{e \in F} \{z(A) : e \subseteq V(A)\} \\ &= (b_v y)(V) + \sum_{A \in \mathcal{W}^-} \frac{b(V(A)) - 1}{2} z(A). \end{aligned}$$

For the last line recall that for both sets  $F$ ,  $F \cap \gamma(V(A))$  is a  $b_x$ -matching for some  $x \in V(A)$ , so  $|F \cap \gamma(V(A))| = \frac{b(V(A)) - 1}{2}$ . We conclude  $w(H_v) = w(F_v)$ , i.e.,  $b\_match$  constructs a maximum  $b_v$ -factor.

To show the algorithm is Las Vegas we only need to observe that the functions  $y, z$  fulfill all the requirements to be optimum linear programming duals. This follows from two more properties. First, the function  $z$  is nonnegative except perhaps on  $\mathcal{V}$ . This follows since  $\zeta^*$  never increases. Second any edge  $e$  of  $G$  satisfies  $w(e) \leq y(e) + z\{A : e \subseteq V(A)\}$ . To see this first note that  $\zeta(e) \leq \zeta(B)$  for  $B$  be the first blossom created with  $e \subseteq V(B)$ . Use this relation to compute  $w(e)$  as in (9).

These two properties make  $y, z$  optimum linear programming duals. So if the properties are satisfied, and the algorithm finds a  $b_v$ -matching composed of edges that satisfy (9), that matching has maximum weight. (This is easily verified without appealing to linear programming, by computing the weight of an arbitrary  $b_v$ -matching  $F$  similar to the computation of  $w(F)$  above.)

To summarize the algorithm of this section works as follows. Consider a  $b$ -critical graph  $G$ , with edge weights  $w(e)$ . Assume we are given the weight of every maximum  $b_v$ -matching, i.e., every value  $w(F_v)$ ,  $v \in V$ . Start by executing the shrinking procedure to construct the tree  $\mathcal{W}$ . For any vertex  $v \in V$ , to find a maximum  $b_v$ -matching call  $b\_match(\mathcal{V}, v)$ .

The total running time for this procedure is  $O(\phi^\omega)$ . To prove this we will show the shrinking procedure uses  $O(m \log n)$  time and  $b\_match$  uses  $O(\phi^\omega)$  time, thus giving the desired bound. First note that  $\mathcal{W}$  has  $\leq 3n$  nodes ( $n$  leaves,  $n$  blossoms that are singletons, and  $n$  larger blossoms).

The shrinking procedure starts by using the given values  $w(F_v)$  to compute the  $\zeta$ -value of each edge. Then it sorts the edges on decreasing  $\zeta$ -value, in time  $O(m \log n)$ .

We use a set-merging algorithm [6] to keep track of the contracted vertices. That is, for any  $v \in V$ ,  $find(v)$  gives the contracted vertex currently containing  $v$ . The operation  $union(A, B, C)$  merges two contracted vertices  $A$  and  $B$  into a new vertex  $C$ .

The iteration of the shrinking procedure that creates  $B$  starts by constructing an adjacency structure for the  $\zeta^*$ -edges of graph  $G_B$ . (An edge  $uv$  ( $u, v \in V$ ) joins the vertices given by  $find(u)$ ,  $find(v)$ .) Then it finds the nontrivial connected components. Each edge of  $G$  is in at most one graph  $G_B$ . So the total time is  $O(m)$  plus the time for  $n$  unions and  $O(m)$  finds [6].

$b\_match$  runs in the time to find  $b_v$ -matchings on all the graphs  $G_B^*$ . These graphs contain a total of  $\leq 3n$  vertices,  $\leq m$  edges, and degree constraints  $b$  totalling  $\leq 3\phi$  (the vertices of  $V$  contribute exactly  $\phi$  and each blossom contributes 1, giving  $\leq 2n + \phi \leq 3\phi$ ). So using an algorithm that finds a  $b$ -factor in time  $O(\phi^\omega)$  gives total running time  $O(\phi^\omega)$ . Since the algorithm is randomized there is a minor point of controlling the error probability when the graphs get small, this is easily handled.

## 6 Shortest paths

This section discusses the single-source shortest-path problem on conservative undirected graphs. It defines a “shortest-path tree” structure for conservative graphs and proves this structure always exists. The structure is a special case of the one for general  $f$ -factors. It is used as an example in Sec.7.

### 6.1 The shortest-path structure

Let  $(G, t, w)$  denote a connected undirected graph with distinguished vertex  $t$  and conservative edge-weight function  $w : E \rightarrow \mathbb{R}$ . We wish to find a shortest path from each vertex to the fixed sink vertex  $t$ .

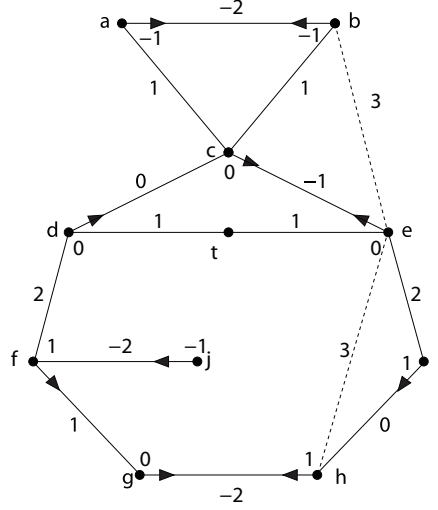


Figure 2: Conservative undirected graph. Vertex labels are shortest-path distances; arrows show the first edge of shortest paths. Dashed edges are not in any shortest path.

Bellman's inequalities needn't hold and a shortest-path tree needn't exist (e.g., the subgraph on  $\{a,b,c\}$  in Fig.2). Nonetheless the distance variables of Bellman's inequalities are optimum for a related set of inequalities.

We begin by defining the analog of the shortest-path tree. When there are no negative edges this analog is a variant of the standard shortest-path tree (node  $\mathcal{V}$  below). Figs.2–3 illustrate the definition. In Fig.2 the arrow from each vertex  $v$  gives the first edge in  $v$ 's shortest path. This edge is  $e(v)$  in the definition below. More generally this edge is  $e(N)$  for any node  $N$  that it leaves, e.g.,  $ce = e(c) = e(\{a, b, c\})$ .

**Definition 25.** A generalized shortest-path tree (gsp-tree)  $\mathcal{T}$  is a tree whose leaves correspond to the vertices of  $G$ . For each node  $N$  of  $\mathcal{T}$ ,  $V(N)$  denotes the set of leaf descendants of  $N$ ,  $V(N) \subseteq V(G)$ . Let  $\mathcal{V}$  be the root of  $\mathcal{T}$ . For each node  $N$ ,  $V(N)$  contains a sink vertex denoted  $t(N)$ ; for  $N \neq \mathcal{V}$ ,  $t(N)$  is the end of an edge  $e(N) \in \delta(V(N))$ . For  $N = \mathcal{V}$  the sink is  $t$ , and we take  $e(\mathcal{V}) = \emptyset$ ; for  $N \neq \mathcal{V}$ ,  $t(N)$  and  $e(N)$  are determined by the parent of  $N$  as described below.

Consider an interior node  $N$  of  $\mathcal{T}$ , with children  $N_i$ ,  $i = 1, \dots, k$ ,  $k \geq 2$ .  $V(N_1)$  contains  $t(N)$  and  $t(N_1) = t(N)$ ,  $e(N_1) = e(N)$ .  $N$  has an associated set of edges  $E(N)$  with  $\{e(N_i) : 1 < i \leq k\} \subseteq E(N) \subseteq \gamma(N)$ . Let  $\overline{N}_i$  denote the contraction of  $V(N_i)$  in  $G$ .

**Case  $N \neq \mathcal{V}$ :**  $E(N)$  forms a (spanning) cycle on the vertices  $\overline{N}_i$ ,  $i = 1, \dots, k$ .

**Case  $N = \mathcal{V}$ :** Either (i)  $E(N)$  gives a cycle exactly as in the previous case, or (ii)  $E(N)$  is a spanning tree on the nodes  $\overline{N}_i$ , rooted at  $N_1$ , with each  $e(N_i)$  the edge from  $N_i$  to its parent.

Note that  $\{e(N_i) : 1 < i \leq k\} = E(N)$  in case (ii) above. In contrast when  $E(N)$  is a cycle, fewer than half its edges may belong to  $\{e(N_i) : 1 < i \leq k\}$ . See edge  $df$  in Fig.2.

For any vertex  $v$ , a top-down traversal of  $\mathcal{T}$  gives a naturally defined  $vt$ -path  $p(v)$  that starts with  $e(v)$ , that we now describe. As an example in Fig.2  $p(j) = j, f, g, h, i, e, c, d, t$ ; in Fig.3 this path is composed of pieces in the subgraphs of 3 nodes,  $j, f$ ;  $f, g, h, i, e$ ; and  $e, c, d, t$ .

For any interior node  $N$  let  $p(v, N) = p(v) \cap \gamma(V(N))$ . So  $p(v) = p(v, \mathcal{V})$ . We will specify  $p(v)$



Since equality holds for  $P = P_v$  we get  $w(P) \geq w(P_v)$  as desired. The generalization to conservative weights will use a laminar family instead of the lower bounds  $2d(x)$ .

A *gsp-structure* consists of a gsp-tree plus two functions  $d, z$ . Each vertex  $v$  has a value  $d(v)$ , its distance to  $t$ . Each node  $N$  of  $\mathcal{T}$  has a value  $z(N)$  that is nonpositive with the exception of  $z(\mathcal{V})$  which has arbitrary sign. Enlarge  $E(G)$  to the set  $E_\ell(G)$  by adding a loop  $xx$  at every vertex except  $t$ , with  $w(xx) = 0$ . Also for any such  $x$  define  $E(x)$  to be  $\{xx\}$  (although  $x$  is a node of  $\mathcal{T}$ , its set  $E(x)$  has not been previously defined); set  $E(t) = \emptyset$ . Say that a node  $N$  of  $\mathcal{T}$  *covers* any edge with both ends in  $N$  (including a loop  $xx$  at  $x \in N - t$ ) as well as the edge  $e(N)$  (if it exists). Every edge  $xy \in E_\ell(G)$  satisfies

$$d(x) + d(y) + w(xy) \geq \sum \{z(N) : N \text{ covers } xy\}, \quad (11)$$

with equality holding for every edge of  $\bigcup \{E(N) \cup e(N) : N \text{ a node of } \mathcal{T}\}$ .

We show this structure guarantees that each path  $p(v)$  is a shortest  $vt$ -path, by an argument similar to the nonnegative case above: Take any  $vt$ -path  $P^-$ . Enlarge it to a spanning subgraph  $P$  by adding the loops  $xx$ ,  $x \notin V(P^-)$ . Similarly define  $P_v$  to be a spanning subgraph formed by adding loops to  $p(v)$ . Let  $\text{cov}(P, N)$  be the number of edges of  $P$  covered by  $N$ . Adding the inequalities (11) for each edge of  $P$  gives

$$d(v) + \sum \{2d(x) : x \in V - v, t\} + d(t) + w(P) \geq \sum \{\text{cov}(P, N)z(N) : N \text{ a node of } \mathcal{T}\} \quad (12)$$

with equality holding for  $P_v$ .

**Claim**  $P_v$  achieves the maximum value of  $\text{cov}(P, N)$ , for every node  $N$ . Furthermore every  $P$  has the same value of  $\text{cov}(P, \mathcal{V})$ .

The claim implies the right-hand side of (12) achieves its minimum when  $P = P_v$  (recall  $z(N) \leq 0$  for every  $N \neq \mathcal{V}$ ). This implies  $w(P) \geq w(P_v)$  as desired.

**Proof of Claim:** Every  $P$  and  $N$  satisfy  $\text{cov}(P, N) \leq |N - t|$ . In proof, each vertex of  $N - t$  is either on a loop of  $P$  or on a maximal subpath of  $P$  from a vertex of  $P$  to an edge of  $\delta(N)$  or to  $t$ .  $\text{cov}(P, N)$  equals  $|N - t|$  decreased by the number of the subpaths that leave  $N$  on an edge  $\neq e(N)$ . This also shows  $\text{cov}(P, N)$  equals  $|N - t|$  for  $P = P_v$  and for  $N = \mathcal{V}$ , as desired  $\diamond$

## 6.2 Basic facts

This section derives the basic structure of shortest-paths to  $t$  for conservative graphs. The key concept, the “planted-cycle”, is essentially a special case of the key concept for  $f$ -factors, the  $2f$ -unifactors. Sec.7 discusses how planted-cycles relate to general  $f$ -factor blossoms (see especially Lemma 28).

For convenience perturb the edge weights by adding  $\epsilon^i$  to the  $i$ th edge. Here  $\epsilon > 0$  is chosen small enough so that no two subsets of  $E_\ell$  have the same weight. Also assume the edges are ordered arbitrarily except that the loops  $xx$  are the last  $n$  edges. Let  $P_v$  denote the unique shortest  $vt$ -path. Let  $F_v$  be its enlargement to a spanning subgraph, i.e.,  $P_v$  plus the loop  $xx$  for every vertex  $x \notin P_v$ . (In the language of  $f$ -factors,  $F_v$  is the unique minimum  $f_v$ -factor, see Sec.7.)

We start with a simple situation where conservative weights do not differ from general nonnegative weights.

**Lemma 26.**  $uv \in P_u - P_v$  implies  $P_u = u, v, P_v$ .

*Proof.* Let  $P_u = u, v, Q_v$ , where  $Q_v$  is a  $vt$ -path, and assume  $Q_v \neq P_v$ . This assumption implies  $w(Q_v) > w(P_v)$ , so the  $ut$ -trail  $T = u, v, P_v$  has  $w(T) < w(P_u)$ , whence  $T$  is not simple. The assumption  $uv \notin P_v$  implies  $T = C, R_u$  for a cycle  $C$  through  $u$  and a  $ut$ -path  $R_u$ .  $w(C) > 0$  implies  $w(R_u) < w(T) < w(P_u)$ , contradiction. ■

This gives a weak analog of the shortest path tree:

**Corollary 27.**  $\{uv : uv \in P_u \oplus P_v\}$  is a forest.

*Proof.* For contradiction let  $x^0, x^1, \dots, x^r$  be a cycle of these edges. Wlog assume  $x^0x^1 \in P_{x^1} - P_{x^0}$ . Lemma 26 shows  $P_{x^1} = x^1, x^0, P_{x^0}$ . This implies  $x^1x^2$  is not in  $P_{x^1}$  so it is in  $P_{x^2} - P_{x^1}$ . Thus  $P_{x^2} = x^2, x^1, x^0, P_{x^0}$ . Continuing this way gives  $P_{x^r} = x^r, x^{r-1}, \dots, x^0, P_{x^0}$ .  $P_{x^r}$  a path implies  $x^r \neq x^0$ , contradiction. ■

We will consider subgraphs of the form  $P_u \cup P_v \cup uv$ , viewed as a multigraph contained in  $2G$ . Define a  $p$ -cycle (“planted-cycle”) to be the union of a cycle  $C$  and 2 copies of a path  $P$  from a vertex  $c \in C$  to  $t$ , with  $V(P) \cap V(C) = \{c\}$ . (Possibly  $c = t$ .)

For motivation first suppose all weights are nonnegative. It is easy to see (e.g., using the shortest-path tree) that

- (a)  $uv \notin P_u \cup P_v$  implies  $P_u \cup P_v \cup uv$  is a  $p$ -cycle.
- (b)  $uv \in P_u \cap P_v$  never occurs (assuming our perturbation of  $w$ ).

In the remaining case  $uv \in P_u - P_v$ , the lemma shows  $P_u \cup P_v \cup uv$  consists of 2 copies of  $P_u$ . This can be viewed as a  $p$ -cycle if we add a loop  $uu$  to the graph. This motivates an approach similar to the algorithm of [14] for matching: Enlarge the graph by adding zero-weight loops  $xx$ ,  $x \in V$ . Define the quantity  $\zeta(uv) = w(P_u) + w(P_v) + w(uv)$ . Repeatedly shrink a cycle of edges with minimum  $\zeta$  value (updating the weights of incident edges in a natural way). The cycles that get contracted are parts of the desired paths  $P_u$ . Once all these pieces are found we can assemble them into the complete paths. (The first edge in  $P_x$  is revealed in the step that shrinks the loop  $xx$ .)

A more precise statement of this “shrinking procedure” is given in Sec.7 (or see [14]); Fig.4 in Sec.7 will show its execution on Fig.2. Once stated, it is a simple exercise to check that the shrinking procedure gets the desired shortest path structure when weights are nonnegative. In fact the shrinking procedure becomes a variant of Dijkstra’s algorithm.

We wish to extend this to conservative weight functions. Observe that in Fig. 2  $P_a \cup P_b \cup ab$  is not a  $p$ -cycle – it contains an extra copy of  $ab$ . We can remedy this by deleting the extra copy. This suggests the following definition:<sup>9</sup>

$$\zeta(uv) = \begin{cases} w(P_u) + w(P_v) + w(uv) & uv \notin P_u \cap P_v \\ w(P_u) + w(P_v) - w(uv) & uv \in P_u \cap P_v. \end{cases} \quad (13)$$

This definition has a similar failure: In Fig. 2  $P_g \cup P_h \cup gh$  contains extra copies of  $gh$ ,  $cd$  and  $ce$ . But we will see this failure is irrelevant to the shrinking procedure, and once again it constructs the desired shortest path structure. The reason is that  $\zeta(gh) = 3$  is not the smallest  $\zeta$  value, and

<sup>9</sup>These expressions are central for general  $f$ -factors – see  $\zeta_{uv}$  and  $\zeta^{uv}$  defined at the start of Sec.8.1.

Fig.4 will show it gets "preempted" by  $\zeta(ce) = 1 < 3$ . (Note also that property (a) fails in Fig. 2:  $P_d \cup P_f \cup df$  contains extra copies of  $cd$  and  $ce$ . Again Fig.4 will show this is irrelevant.)

For the rest of this section assume

$$|\delta(t)| = 1.$$

(This can always be achieved by adding a dummy edge incident to  $t$ . The shortest-path structure for the given graph can easily be derived from the structure for the enlarged graph.) The assumption implies any p-cycle has  $c \neq t$ .

Let  $cc'$  be the multiplicity 2 edge incident to  $C$ . We use the notation  $C, c, c'$  throughout the discussion. When convenient we do not distinguish between the p-cycle, or  $C$ , or the pairs  $C, c$  or  $C, cc'$ . Note that each vertex  $x \in C - c$  has exactly 2  $xt$ -paths in the p-cycle, depending on which direction we traverse  $C$ .

The following lemma gives properties of p-cycles that are shared with general blossoms, as well as properties that are specific to shortest paths (see Sec.7).

Let  $E^*$  be the set of edges in  $\{uv : uv \notin P_u \oplus P_v\}$  whose  $\zeta$ -value is smallest, and let  $\zeta^*$  be this smallest  $\zeta$ -value.

**Lemma 28.** (i) *There is a p-cycle whose cycle  $C$  consists of  $E^*$  and possibly other edges of  $\zeta$ -value  $\leq \zeta^*$ .*

(ii) *For every  $x \in C$ ,  $P_x$  is one of the  $xt$ -paths in the p-cycle. In particular  $P_c$  starts with edge  $cc'$ .*

(iii) *Every  $x \in C$  has  $2w(P_x) \leq \zeta^*$ .*

*Proof.* First note the lemma is trivial if  $E^*$  consists of a loop, so suppose not. The bulk of the argument consists of several claims that lay the foundation for (i)–(iii).

Take any edge  $uv \in E^*$ . Consider the subgraph  $S$  of  $2G$  formed by modifying  $P_u \cup P_v$  to contain exactly one copy of  $uv$ . Note that

$$w(S) = \zeta(uv) = \zeta^*. \quad (14)$$

We will show  $S$  is the desired p-cycle.

**Claim 1:**  *$E(S)$  can be partitioned into*

- (a) *a circuit  $C = (P_u \oplus P_v) + uv$ ;*
- (b) *a multiplicity 2 path  $P$  from  $t$  to some  $c \in C$  ( $V(P) \cap V(C) = \{c\}$ );*
- (c) *zero or more multiplicity 2 paths joining two distinct vertices of  $C - c$ .*

**Proof:** We start by analyzing the multiplicity 1 edges of  $S$ . We work in the enlarged graph with edges  $E_\ell(G)$ . Let  $F_u \subseteq E_\ell(G)$  be  $P_u$  enlarged to an  $f_u$ -factor by adding loops; similarly for  $F_v$ . The subgraph  $F_u \oplus F_v$  contains a  $uv$ -trail  $T$  that starts with an edge of  $F_v - F_u$  incident to  $u$ , ends with an edge of  $F_u - F_v$  incident to  $v$ , and has edges alternating between  $F_v - F_u$  and  $F_u - F_v$ . (Any of these edges including the first and last may be loops.)  $F_u \oplus T$  is an  $f_v$ -factor, so

$$w(F_u) - w(F_u \cap T) + w(F_v \cap T) \leq w(F_v).$$

Similarly

$$w(F_v) - w(F_v \cap T) + w(F_u \cap T) \leq w(F_u).$$

Adding these inequalities gives  $w(F_u) + w(F_v) \leq w(F_v) + w(F_u)$ . Thus all inequalities hold with equality. The perturbed weight function implies  $F_u - F_u \cap T + F_v \cap T = F_v$ . In particular  $F_u - T =$

$F_v - T$ . So the nonloop edges in  $T + uv$  are precisely the multiplicity 1 edges in  $S$ . Also we get (a) of the Claim.

If  $T$  contains a loop  $xx$ ,  $x \neq u, v$ , then the 2 edges that immediately precede and follow  $xx$  in  $T$  are both in  $P_u - P_v$  or both in  $P_v - P_u$ . This implies  $P_u \oplus P_v$  is a trail consisting of 1 or more subpaths, each of 1 or more edges, that alternate between  $P_u - P_v$  and  $P_v - P_u$ .

$S$  contains a multiplicity 2 path from  $t$  to some  $c \in T$  (with no other vertex of  $T$ ). We get (b) of the Claim.

Let  $X$  be the set of all vertices of  $T - c$  that are on an edge of  $P_u \cap P_v - uv$ . If  $x \neq u, v$  is the end of an edge  $P_u \cap P_v - uv$  then  $x$  is on another edge of  $P_u$  and another edge of  $P_v$ . It is easy to see this implies that each vertex of  $X$  is joined to another vertex of  $X$  by a multiplicity 2 path of  $S$ . These are the paths of (c) of the Claim.

Finally note that (a)–(c) account for all edges of  $S$ : An edge of  $P_u \oplus P_v$  is in  $C$ . An edge of  $P_u \cap P_v$  is in a path or cycle of such edges. A path either has both ends in  $C$  (making it type (c)) or one end in  $C$  (making the other end  $t$ , so the path is type (b)). A cycle of  $P_u \cap P_v$  edges cannot exist since  $P_u$  is acyclic.  $\diamond$

The above  $X$  is a set, not a multiset, i.e., a vertex  $x \in X$  is the end of only 1 type (c) path. This follows from  $d(x, S) = 4$ , which also holds if  $x = c$ . It is worthwhile to describe the case  $x = u$ : If  $uv \in P_u \cap P_v$  then  $d(u, S) = 2$ , so  $u \notin X$ . Suppose  $uv \notin P_u \cap P_v$  and  $u \in X$ . The first edge of  $P_u$ , say  $f$ , must belong to  $P_v$ ; let  $g$  be the other edge of  $P_v \cap \delta(u)$ . Then  $S \cap \delta(u)$  consists of 2 copies of  $f$  plus the edges  $g, uv$ .

**Claim 2:** *The edges of type (a) and (c) can be partitioned into a collection of cycles.*

**Proof:** Any circuit is the edge-disjoint union of cycles. So we can assume there are type (c) edges, i.e., the above set  $X$  is nonempty.

$|X|$  is even, so the vertices of  $X$  divide the edges of  $C$  into an even number of segments  $C_i$ . Partition the edges of  $C$  into 2 sets  $\mathcal{C}_1, \mathcal{C}_2$ , each consisting of alternate segments  $C_i$ . Partition the edges of the type (c) paths into 2 sets  $\mathcal{P}_1, \mathcal{P}_2$ , each consisting of 1 copy of each type (c) path  $P_j$ . Now the edges of type (a) and (c) are partitioned into the two sets nonempty  $\mathcal{P}_s \cup \mathcal{C}_s$ ,  $s = 1, 2$ .

Let  $\mathcal{PC}$  be one of the subgraphs  $\mathcal{P}_s \cup \mathcal{C}_s$ . Observe that each vertex  $x$  has  $d(x, \mathcal{PC})$  even: If  $x$  is interior to a  $P_j$  then  $d(x, \mathcal{PC}) = 2$ . If  $x$  is the end of a  $P_j$  then it is the end of a corresponding  $C_i$ , so again  $d(x, \mathcal{PC}) = 2$ . (Recall that  $X$  is not a multiset!) Any other  $x$  has 1 or 2 occurrences in  $\mathcal{PC}$ , both interior to  $C_i$ 's. (2 occurrences may correspond to  $x$  occurring twice in some  $C_i$ , or once in two different  $C_i$ 's.) Thus  $d(x, \mathcal{PC}) \in \{2, 4\}$ .

We conclude that each connected component of  $\mathcal{PC}$  is a circuit (the construction ensures the circuit is edge-simple, since any edge of  $G$  occurs at most once in  $\mathcal{PC}$ ). So  $\mathcal{PC}$  is a union of cycles.  $\diamond$

**Claim 3:**  *$C$  is a cycle and there are no type (c) edges.*

**Proof:** Suppose the partition of Claim 2 consists of a single cycle. Then there are no type (c) edges (since each of the above sets  $\mathcal{P}_s \cup \mathcal{C}_s$  is nonempty). So Claim 3 holds.

Now for the purpose of contradiction assume the partition of Claim 2 contains at least 2 cycles. Each cycle has nonnegative weight. In fact the perturbation implies the weight is positive.

One of these cycles, call it  $B$ , contains vertex  $c$ . Form a subgraph  $S'$  by using the cycle  $B$  and the type (b) path of  $S$  from  $t$  to  $c$ . Our assumption implies  $w(S') < w(S)$ .

Let  $B$  contain an edge  $xy \notin P_x \oplus P_y$  (Corollary 27). Let  $B_x$  ( $B'_x$ ) be the  $xt$ -trail contained in  $S'$

that avoids (contains)  $xy$ , respectively. Define  $B_y$  and  $B'_y$  similarly. Then

$$w(B_x) + w(B_y) + w(xy) = w(B'_x) + w(B'_y) - w(xy) = w(S') < w(S) = \zeta^*. \quad (15)$$

(The last equation is (14).) The definition of  $\zeta(xy)$  shows it is at most either the first expression of (15) (if  $xy \notin P_x \cup P_y$ ) or the second expression (if  $xy \in P_x \cap P_y$ ). Thus  $\zeta(xy) < \zeta^*$ , contradiction.  $\diamond$

Now we prove (i)–(iii). Take any edge  $xy \in C$ . The relation (15) becomes  $w(B_x) + w(B_y) + w(xy) = w(B'_x) + w(B'_y) - w(xy) = w(S) = \zeta^*$ . So  $\zeta(xy) \leq \zeta^*$ . If  $xy \notin P_x \oplus P_y$  we get  $\zeta(xy) = \zeta^*$ , and  $P_x, P_y$  is either  $B_x, B_y$  or  $B'_x, B'_y$ , i.e., (ii) holds for  $x$ .

To prove (i) it remains only to show  $C$  contains every edge  $u'v' \in E^*$ . Analogous to (14), the subgraph  $S'$  formed from  $u'v'$  the same way  $S$  is formed from  $uv$  has weight  $w(S') = \zeta(u'v') = \zeta^*$ . The perturbed edge weight function implies  $S = S'$ . Thus exactly 1 copy of  $u'v'$  belongs to  $S$ , i.e.,  $u'v' \in C$ .

To prove the first assertion of (ii) we need only treat the case  $xy \in P_x \oplus P_y$ . (Note the second assertion of (ii) is a simple special case of the first.) These edges are a proper subset of  $C$  (Corollary 27). Let  $Q$  be a maximal length path of such edges that does not contain  $c$  internally. At least one end of  $Q$ , say  $r$ , is on an edge  $rs \in C - (P_r \oplus P_s)$  (the other end may be  $c$ ). If  $rs \notin P_r \cup P_s$  then  $P_r$  is the  $rt$ -path that avoids  $rs$  in  $C$ . Lemma 26 shows any  $x \in Q$  has  $P_x$  a subpath of  $P_r$ . Thus (ii) holds for  $x$ . Similarly if  $rs \in P_r \cap P_s$  then  $P_r$  is the  $rt$ -path that contains  $rs$  in  $C$ , and Lemma 26 shows any  $x \in Q$  has  $P_x$  the  $xt$ -subpath of  $C$  containing  $rs$ . Again (ii) holds for  $x$ .

To prove (iii) first assume  $x \neq c$ . Thus  $\zeta^* = w(S) = w(B_x) + w(B'_x) > 2w(P_x)$ . For  $x = c$  the argument is similar: Since  $G$  is conservative,  $\zeta^* = w(S) = 2w(P_c) + w(C) > 2w(P_c)$ . ■

We want the above cycle  $C$  to be a cycle node in the gsp-tree. Lemma 28(ii) shows  $C$  has the required properties for vertices  $x \in C$ . Now we show  $C$  has the required properties for  $x \notin C$ .

A vertex  $x$  *respects* a p-cycle if either  $x \in C$  and  $P_x$  is an  $xt$ -path in the p-cycle, or  $x \notin C$  and  $P_x$  either contains no vertex of  $C$  or it contains exactly the same edges of  $\gamma(C) + cc'$  as some  $P_y$ ,  $y \in C$ .

Let  $C, cc'$  be the p-cycle of Lemma 28.

**Lemma 29.** *Every vertex  $x$  respects  $C$ .*

**Remark:** We allow  $C$  to be a loop  $cc \in E_\ell(G)$ . In this case the lemma states that a shortest path  $P_x$  that contains  $c$  actually contains  $cc'$ .

*Proof.* Lemma 28(ii) shows we can assume  $x \notin C$ . The argument begins similar to Claim 1 of Lemma 28. We work in the graph with edges  $E_\ell(G)$ . For any vertex  $u$  let  $F_u \subseteq E_\ell(G)$  be  $P_u$  enlarged to an  $f_u$ -factor. The subgraph  $F_x \oplus F_c$  contains an  $xc$ -trail that starts with an edge of  $\delta(x) \cap F_c - F_x$ , ends with an edge of  $\delta(c) \cap F_x - F_c$ , and has edges alternating between  $F_c - F_x$  and  $F_x - F_c$ . Let  $e$  be the first edge of the trail that belongs to  $\delta(C)$ . Let  $T$  be the subtrail that starts at  $x$  and ends with edge  $e$ .

Consider two cases:

**Case  $e \neq cc'$ :** Clearly  $e \in F_x - F_c$ . Let  $e$  be incident to vertex  $a \in C$ .  $T$  is a subgraph of  $F_x \oplus F_a$  (Lemma 28(ii)). Now follow the argument of Lemma 28 Claim 1:  $F_x \oplus T$  is an  $f_a$ -factor

and  $F_a \oplus T$  is an  $f_x$ -factor, so we get  $F_x - F_x \cap T + F_a \cap T = F_a$  and  $F_x - T = F_a - T$ . The latter implies  $F_x$  and  $F_a$  contain the same edges of  $\gamma(C) + cc'$ , i.e.,  $x$  respects  $C$ .

**Case  $e = cc'$ :** Clearly  $cc' \in F_c - F_x$ . Define a degree-constraint function  $f^c$  by

$$f^c(x) = \begin{cases} 1 & x = t \\ 3 & x = c \\ 2 & x \neq t, c. \end{cases}$$

An  $f^c$ -factor (in  $E_\ell(G)$ ) consists of a  $ct$ -path plus a cycle through  $c$ , plus loops at the remaining vertices. Since  $G$  is conservative and  $w$  has been perturbed,  $cc$  is the smallest cycle through  $c$ . So  $F^c$ , the minimum-weight  $f^c$ -factor, consists of  $P_c$  plus a loop at every vertex except  $t$  (in particular  $cc \in F^c$ ). Thus  $F^c = F_c + cc$ .

Now the argument follows the previous case:  $T$  is a subgraph of  $F_x \oplus F^c$ .  $F_x \oplus T$  is an  $f^c$ -factor,  $F^c \oplus T$  is an  $f_x$ -factor, so  $F_x - F_x \cap T + F_c \cap T = F^c$  and  $F_x - T = F^c - T$ . We have already noted  $cc' \notin F_x$  and the last equation shows  $F_x$  contains the same edges of  $\gamma(C)$  as  $F^c$ , i.e., every loop  $aa, a \in C$ . In other words  $P_x$  does not contain a vertex of  $C$ , so  $x$  respects  $C$ . ■

### 6.3 Construction of the shortest-path structure

The last two lemmas show how to construct the first node of the gsp-tree. We construct the remaining nodes by iterating the procedure. This section first shows how to construct the gsp-tree; then it completes the gsp-structure by constructing  $z$ .

For the gsp-tree we first state the algorithm and then prove its correctness. *Shrinking a p-cycle* means contracting its cycle  $C$ ;  $C$  is the *shrunk cycle*. Let  $\overline{G}$  be a graph formed by starting with  $G$  and repeatedly shrinking a p-cycle. (So the collection of shrunk cycles forms a laminar family.) We treat  $\overline{G}$  as a multigraph that contains parallel edges but not loops. It is convenient to refer to vertices and edges of  $\overline{G}$  by indicating the corresponding objects in  $G$ . So let  $\overline{E}(G)$  denote the set of edges of  $G$  that correspond to (nonloop) edges in  $\overline{G}$ . Thus writing  $xy \in \overline{E}(G)$  implies  $x, y \in V(G)$ . We do not distinguish between  $xy$  and its image in  $\overline{G}$ . Similarly, writing  $C, cc'$  for a shrunk p-cycle implies  $cc' \in \overline{E}(G)$ , and a  $vt$ -path in  $\overline{G}$  has  $v \in V(G)$  and its first edge incident to  $v$ . An overline denotes quantities in  $\overline{G}$ , e.g.,  $\overline{w}, \overline{\zeta}$ .

The following algorithm constructs the gsp-tree  $\mathcal{T}$ . We assume the shortest paths  $P_x$  are known. (The function  $z$  is constructed below.)

Initialize  $\overline{G}$  (the current graph) to the graph  $(V, E_\ell(G))$ , and  $\mathcal{T}$  to contain each vertex of  $G$  as a singleton subtree. Then repeat the following step until  $\overline{G}$  is acyclic:

Let  $C, cc'$  be the p-cycle of weight  $\zeta^*(\overline{G})$  given by Lemma 28. Shrink  $C$  in  $\overline{G}$ . Set  $e(C) = cc'$ .

Unless  $C$  is a loop, create a node in  $\mathcal{T}$  whose children correspond to the vertices of  $C$ .

When the loop halts create a root node of  $\mathcal{T}$  whose children correspond to the vertices of the final graph  $\overline{G}$ .<sup>10</sup>

To complete the description of this algorithm we must specify the weight function  $\overline{w}$  for  $\overline{G}$ . Let  $\mathcal{C}$  be the collection of maximal shrunk cycles that formed  $\overline{G}$ . An edge  $cc' \in \overline{E}(G)$  may be associated

<sup>10</sup>In contrast with the general definition, our assumption that  $t$  is on a unique edge ensures the root of  $\mathcal{T}$  is always a tree node.

with two cycles of  $\mathcal{C}$ , one at each end. For a p-cycle  $C, cc'$  in  $\mathcal{C}$ , as in Definition 25  $V(C)$  denotes the set of vertices of  $G$  that belong to  $C$  or a contracted vertex of  $C$ .  $V(\mathcal{C})$  is the union of all the  $V(C)$  sets. For  $C, c \in \mathcal{C}$  and  $x \in V(C)$ ,  $C_x \subseteq E(G)$  denotes the minimum-weight  $xc$ -path contained in  $\gamma(C, G)$ .

Let  $G$  denote the given graph, and let

$$\omega = 2|w|(E).$$

The weight of an edge  $e = xy \in \overline{E}(G)$  in  $\overline{G}$  is defined to be  $\overline{w}(e) = w(e) + \Delta(e, x) + \Delta(e, y)$ , where

$$\Delta(e, x) = \begin{cases} 0 & x \notin V(\mathcal{C}) \\ -2\omega & \mathcal{C} \text{ contains p-cycle } C, e \text{ with } x \in V(C) \\ 2\omega + w(C_x) & \mathcal{C} \text{ contains p-cycle } C, cc' \text{ with } x \in V(C), cc' \neq e. \end{cases}$$

We will use a variant of the “respects” relation. Let  $\overline{S}$  be a cycle or path in  $\overline{G}$ . Let  $C, cc'$  be a p-cycle of  $\mathcal{C}$ , and  $\overline{C}$  the contracted vertex for  $C$  in  $\overline{G}$ .  $\overline{S}$  *respects*  $\overline{C}$  if  $\overline{S} \cap \delta(\overline{C})$  is either empty or consists of  $cc'$  plus  $\leq 1$  other edge. This definition corresponds to the previous definition if we view  $\overline{C}$  as a p-cycle whose cycle is a loop  $\overline{C}\overline{C}$ . More importantly we shall use this fact: If a path  $P$  respects  $C, cc'$  and  $\overline{G}$  is derived from  $G$  by contracting  $C$  to a vertex  $\overline{C}$ , then the image of  $P$  in  $\overline{G}$  respects  $\overline{C}$ . Also, for any  $C, cc'$  in  $\mathcal{C}$ , if the image of a shortest path  $P_u$  respects  $\overline{C}$  and contains  $\overline{C}$  internally, then  $P_u$  traverses  $C$  along the path  $C_x$  that joins the 2 edges of  $P_x \cap \delta(C)$ . This follows from the optimality of  $P_u$ .

$\overline{S}$  *respects*  $\overline{C}$  if it respects each  $\overline{C} \in \mathcal{C}$ . When  $\overline{S}$  respects  $\overline{C}$ , the *preimage* of  $\overline{S}$  is the subgraph  $S$  of  $G$  that completes  $\overline{S}$  with minimum weight, i.e.,  $S$  consists of the edges of  $\overline{S}$  plus, for each p-cycle  $\overline{C}$  on 2 edges of  $\overline{S}$ , the minimum-weight path  $C_x$  in  $C$  that joins the 2 edges. This preimage is unique. (Any  $C, cc'$  of  $\mathcal{C}$  that is on 2 edges of  $\overline{S}$  is on  $cc'$  and another edge that determines the vertex  $x$  in the definition of the preimage. Note that if  $\overline{S}$  is a  $vt$ -path that respects  $\overline{C}$  and  $v \in V(\mathcal{C})$  then  $v = c$  for some p-cycle  $C, cc'$  of  $\mathcal{C}$ ; so in this case the preimage of  $\overline{S}$  does not contain any edges of  $C$ .)

Lemma 32 below shows the following properties always hold.

P1:  $\overline{G}$  is conservative.

P2: For any vertex  $x \notin V(\mathcal{C})$ ,  $P_x$  is the preimage of  $\overline{P}_x$ . For any p-cycle  $C, cc' \in \mathcal{C}$ ,  $P_c$  is the preimage of  $\overline{P}_{\overline{C}}$ , both of which start with edge  $cc'$ . For any other  $x \in V(C)$ , the image of  $P_x$  in  $\overline{G}$  is  $\overline{P}_{\overline{C}}$ .

**Lemma 30.** *Assuming P1–P2 always hold,  $\mathcal{T}$  is a valid gsp-tree for  $G$ .*

*Proof.* First observe that every vertex  $x \in V(G)$  gets assigned a value  $e(x)$ . This holds as long as some iteration chooses  $xx$  as the minimum weight p-cycle. So for the purpose of contradiction, suppose an iteration contracts a p-cycle  $C$  where  $x \in C$  but the loop  $xx$  has not been contracted. Since  $xx \notin P_x \oplus P_x = \emptyset$ ,  $xx$  is considered in the definition of  $E^*$ . So we get the desired contradiction by proving

$$\zeta(xx) = w(xx) + 2w(P_x) < \zeta^*. \quad (16)$$

Lemma 28(iii) shows  $2w(P_x) \leq \zeta^*$  (in any iteration). We can assume the perturbation of  $w$  gives every subgraph of  $2G$  a distinct weight. Thus  $2w(P_x) < \zeta^*$ . Furthermore we can assume the

perturbation enforces a lexical ordering of the edges. Since the loop  $xx$  is ordered after any edge of  $G$ , it cannot reverse this inequality, i.e., (16) holds.

The algorithm sets  $e(C)$  correctly for each node  $C$  of  $\mathcal{T}$ , by definition. Recall the path  $p(v, N)$  from (10). It is easy to see the lemma amounts to proving that for any interior node  $C$  of  $\mathcal{T}$  and any vertex  $x \in V(C)$ , the edges of  $E(P_x) \cap \gamma(C)$  correspond to  $p(x, C)$  as defined in (10). Let p-cycle  $B, bb'$  be the child of  $C$  with  $x \in B$ . ( $B$  is a loop when the child of  $C$  is a leaf of  $\mathcal{T}$ .) We consider two similar cases for node  $C$ .

Suppose  $C, cc'$  is a cycle node of  $\mathcal{T}$ . Let  $\overline{G}$  be the graph immediately before  $C$  is contracted. P2 in  $\overline{G}$  shows  $P_b$  is the preimage of  $P_{\overline{B}}$ . It also shows  $P_{\overline{B}}$  starts with  $bb'$ . Lemma 28(ii) shows  $P_{\overline{B}}$  is one of the  $\overline{B}$ -paths in the p-cycle. We have already observed that  $C$  and its children have the correct  $e$ -values. Thus  $\mathcal{T}$  specifies the desired path  $p(b, C)$  as defined in (10). This extends to any  $x \in V(B)$  by the last assertion of P2.

The remaining case is for the root node  $\mathcal{V}$  of  $\mathcal{T}$ , with  $\mathcal{V}$  a tree node. In the final acyclic graph, P2 shows  $P_b$  is the preimage of  $P_{\overline{B}}$ . Thus  $\mathcal{T}$  specifies the desired path  $p(b, \mathcal{V})$ . As before this extends to any vertex  $x \in V(B)$ .  $\diamond \blacksquare$

We complete the gsp-structure by specifying  $z$ . For each node  $N$  of  $\mathcal{T}$  let  $\zeta_N$  be the weight  $\zeta^*(\overline{G})$  of its corresponding p-cycle. For a leaf  $x$  the corresponding p-cycle is the loop  $xx$ ; for the root  $\mathcal{V}$  of  $\mathcal{T}$ , which is a tree node, the corresponding p-cycle is the last p-cycle to get shrunk. Let  $p$  be the parent function in  $\mathcal{T}$ . For each node  $N$  define

$$z(N) = \begin{cases} \zeta_N & N = \mathcal{V} \\ \zeta_N - \zeta_{p(N)} & N \neq \mathcal{V} \end{cases}$$

Lemma 32 shows this additional property always holds:

P3: For any  $uv \in \overline{E}(G)$ ,

$$\overline{\zeta}(uv) = \begin{cases} \zeta(uv) & uv \notin P_u \oplus P_v \\ \zeta(uv) & uv \in P_u - P_v, u \notin V(\mathcal{C}) \\ \zeta(uv) - 4\omega & uv \in P_u - P_v, u \in V(\mathcal{C}). \end{cases}$$

The notation in P3 is unambiguous since  $uv \in P_u$  iff  $uv \in P_{\overline{u}}$  by P2.

**Lemma 31.** *Assuming P1–P3 always hold,  $\mathcal{T}$  with the above function  $z$  is a gsp-structure for  $G$ .*

*Proof.* The definition of  $z$  clearly implies that for any node  $N$  of  $\mathcal{T}$ , the  $z$ -values of all the ancestors of  $N$  (including  $N$ ) sum to  $\zeta_N$ . We claim  $\zeta^*(\overline{G})$  increases every iteration. The claim implies that for every  $N \neq \mathcal{V}$ ,  $z(N) = \zeta_N - \zeta_{p(N)} \leq 0$ , i.e.,  $z(N)$  is nonpositive as desired. To prove the claim note that for a fixed edge  $uv$ ,  $uv$  belongs to  $P_u \oplus P_v$  in one iteration iff it does in the next iteration, as long as it is not contracted (by P2). Hence the only change in  $E^*$  from one iteration to the next is that contracted edges leave  $E^*$ . P3 shows the edges in  $E^*$  retain their original  $\zeta$ -values. Thus  $\zeta^*(\overline{G})$  never decreases.

To complete the proof we must show (11) for every  $uv \in E_\ell(G)$ , with equality for  $e(N)$  edges and other edges of subgraphs  $E(N)$ . Consider two cases.

**Case**  $uv \notin P_u \cap P_v$ : By definition

$$d(u) + d(v) + w(uv) = \zeta(uv).$$

Let  $N$  be the deepest node of  $\mathcal{T}$  that covers  $uv$ . Observe that

$$\zeta(uv) \geq \zeta_N,$$

with equality holding if  $uv \in E(N) \cup e(N)$ . In proof, if  $uv \notin P_u \cup P_v$  this follows from  $uv \in E^*$ . If  $uv \in P_u - P_v$ , then Lemma 26 shows  $uv = e(u)$ . So the loop  $uu$  covers  $uv$ ,  $N = \{u\}$ , and equality holds. Since the sum of the right-hand side of (11) equals  $\zeta_N$ , combining the two displayed relations gives the desired conclusion for (11).

**Case**  $uv \in P_u \cap P_v$ : By definition

$$d(u) + d(v) - w(uv) = \zeta(uv).$$

$uv$  is covered by the nodes of  $\mathcal{T}$  that are ancestors of  $u$  or  $v$ . The ancestors of  $u$  have  $z$ -values summing to  $\zeta(uu) = 2d(u)$  and similarly for  $v$ . Let node  $A$  be the least common ancestor of  $u$  and  $v$  in  $\mathcal{T}$ . Since  $uv$  is in the cycle of node  $A$ ,  $\zeta_A = \zeta(uv) = d(u) + d(v) - w(uv)$  (by definition of  $\zeta$ ). The sum of the right-hand side of (11) equals

$$\zeta_u + \zeta_v - \zeta_A = 2d(u) + 2d(v) - (d(u) + d(v) - w(uv)) = d(u) + d(v) + w(uv)$$

as desired. ■

The development is completed by establishing P1–P3:

**Lemma 32.** *P1–P3 hold in every iteration.*

*Proof.* Consider two edges  $e_i = x_i y_i \in \overline{E}(G)$ ,  $i = 1, 2$ , where both  $x_i$  have the same image in  $\overline{G}$ . If  $x_i \notin V(\mathcal{C})$  then

$$\Delta(e_1, x_1) + \Delta(e_2, x_2) = 0. \quad (17)$$

Suppose the  $x_i$  belong to the p-cycle  $C$  of  $\mathcal{C}$ . If  $e_1, e_2$  respects  $\overline{C}$  then wlog the p-cycle corresponds to  $C, e_2$ , and

$$\Delta(e_1, x_1) + \Delta(e_2, x_2) = w(C_{x_1}). \quad (18)$$

In the remaining case, i.e.,  $e_1, e_2$  does not respect  $\overline{C}$ ,

$$\Delta(e_1, x_1) + \Delta(e_2, x_2) \geq 2(\omega + |w|(E - \gamma(C))). \quad (19)$$

**Claim 1:** *Let  $\overline{S}$  be a cycle of  $\overline{G}$ . If  $\overline{S}$  respects  $\overline{C}$  then  $\overline{w}(\overline{S}) = w(S)$  for  $S$  the preimage of  $\overline{S}$ . If  $\overline{S}$  does not respect  $\overline{C}$  then  $\overline{w}(\overline{S}) \geq 2\omega$ .*

**Proof:** Let  $\mathcal{CR}$  ( $\mathcal{CN}$ ) contain the p-cycles of  $\mathcal{C}$  that are respected (not respected) by  $\overline{S}$ , respectively. Let  $CX = \bigcup \{C_x : x = x_1 \text{ in (18) for } C \in \mathcal{CR}\}$ . Then (17)–(19) imply

$$\overline{w}(\overline{S}) \geq w(\{e : e \in \overline{S} \cup CX\}) + \sum \{2(\omega + |w|(E - \gamma(C))) : C \in \mathcal{CN}\}. \quad (20)$$

Here we use the fact that the  $2\omega$  terms cancel on the 2 edges incident to a cycle of  $\mathcal{CR}$  (this property actually depends on our assumption that  $t$  has a unique incident edge).

When  $\overline{S}$  respects  $\overline{\mathcal{C}}$ , i.e.,  $\mathcal{CN} = \emptyset$ , (20) holds with equality, and we get the claim. Suppose  $\overline{S}$  does not respect  $\overline{\mathcal{C}}$ . Then choosing  $B$  as any p-cycle of  $\mathcal{CN}$ , (20) implies  $\overline{w}(\overline{S}) \geq w(\{e : e \in \overline{S} \cup CX\}) + 2(\omega + |w|(E - \gamma(B))) \geq 2\omega$ , giving the claim.  $\diamond$

Claim 1 implies property P1.

**Claim 2:** Let  $\overline{S}$  be a  $vt$ -path in  $\overline{G}$ . If  $\overline{S}$  does not respect  $\overline{\mathcal{C}}$  then  $\overline{w}(\overline{S}) \geq \omega$ . If  $\overline{S}$  respects  $\overline{\mathcal{C}}$  let  $S$  be the preimage of  $\overline{S}$ . If  $v \notin V(\mathcal{C})$  then  $\overline{w}(\overline{S}) = w(S)$ . If  $v \in V(\mathcal{C})$  then  $\overline{w}(\overline{S}) = w(S) - 2\omega$ .

**Proof:**  $\overline{S}$  satisfies a version of (20) that accounts for the term  $\Delta(e_1, v)$  for edge  $e_1 = \delta(v, \overline{S})$ . We examine several cases.

Suppose  $\overline{S}$  respects  $\overline{\mathcal{C}}$ . If  $v \notin V(\mathcal{C})$  then (20) holds with equality, giving the claim. If  $v \in V(\mathcal{C})$  then  $\mathcal{C}$  contains a p-cycle  $C, e_1$  with  $v \in V(C)$ .  $\overline{w}(\overline{S})$  contains an extra term  $\Delta(e_1, v) = -2\omega$ , again giving the claim.

Suppose  $\overline{S}$  does not respect  $\overline{\mathcal{C}}$ . If  $v \notin V(\mathcal{C})$  then (20) holds unmodified. As in Claim 1,  $\overline{w}(\overline{S}) \geq 2\omega$ , giving the current claim. So suppose  $v \in V(B)$  for  $B \in \mathcal{C}$ . If  $e_1$  does not respect  $\overline{B}$   $\overline{w}(\overline{S})$  contains an extra term  $\Delta(e_1, v) = 2\omega + w(B_v) \geq \omega + |w|(E - \gamma(B_v))$ , giving the claim. In the remaining case  $e_1$  respects  $\overline{B}$  and  $\overline{S}$  does not respect some  $\overline{A} \neq \overline{B}$ . The right-hand side of (20) contains the extra term  $\Delta(e_1, v) = -2\omega$ , and the term for  $A$  is at least  $3\omega + 2|w|(E - \gamma(A))$ . These two contributions sum to  $\geq \omega + |w|(E - \gamma(A))$ . Thus the right-hand side of (20) is  $\geq \omega$ , as desired.  $\diamond$

**Claim 3:** P2 holds every iteration.

We argue by induction on the number of iterations. Consider an iteration for the p-cycle  $C, cc'$ . Let  $\overline{G}^-, \overline{w}^- (\overline{G}^+, \overline{w}^+)$  be the graph and weight function immediately before (after)  $C$  is contracted, respectively. For greater precision, if  $H$  is  $\overline{G}^-$  or  $\overline{G}^+$  and  $z$  is a vertex of  $H$ , let  $P(z, H)$  denote the shortest  $zt$ -path in  $H$ . Take any  $x \in V(G)$ . Let  $P = P(\overline{x}, \overline{G}^+)$ . Note that  $P_x$  continues to denote the shortest  $xt$ -path in  $G$ , and for P2 we want to establish the relationship between  $P$  and  $P_x$ .  $\mathcal{C}$  denotes the family of contracted vertices in  $\overline{G}^+$ , i.e., it includes  $C$ .

First assume either  $x \notin V(\mathcal{C})$  or  $x = b$  for some p-cycle  $B, bb'$  of  $\mathcal{C}$  ( $x = c$  is a possibility). Let  $\delta$  be 0 ( $2\omega$ ) if  $x \notin V(\mathcal{C})$  ( $x = b$ ) respectively. Let  $Q = P(\overline{x}, \overline{G}^-)$ . Lemma 29 shows  $Q$  respects  $\mathcal{C}$ . Thus  $Q$  has an image  $Q^+$  in  $\overline{G}^+$  that respects  $\overline{\mathcal{C}}$ . The inductive assumption of P2 in  $\overline{G}^-$  shows  $P_x$  is the preimage of  $Q$ . (For  $x = c$ ,  $\overline{x}$  in  $\overline{G}^-$  may be a vertex or a contracted cycle, and we use the appropriate assertion of P2.) Thus the optimality of  $P_x$  implies it is the preimage of  $Q^+$ . Thus Claim 2 shows

$$\overline{w}^-(Q) = \overline{w}^+(Q^+) = w(P_x) - \delta.$$

Since  $\omega > w(P_x)$ , Claim 2 shows  $P$  respects  $\overline{\mathcal{C}}$  and

$$\overline{w}^+(P) = w(P^-) - \delta$$

for  $P^-$  the preimage of  $P$ . Since  $P^-$  is an  $xt$ -path,

$$w(P^-) \geq w(P_x).$$

Combining the inequalities gives  $\overline{w}^+(Q^+) \leq \overline{w}^+(P)$ . Thus  $Q^+ = P$ . So as asserted by the first part of P2,  $P_x$  is the preimage of  $P_x$ .

Also, as in the second assertion of P2, every p-cycle  $B, bb'$  of  $\mathcal{C}$  has  $P_b$  and  $P(\overline{B}, \overline{G}^+) = P(\overline{b}, \overline{G}^+)$  both starting with edge  $bb'$ . This follows since  $P(\overline{b}, \overline{G}^-)$  starts with  $bb'$  (by definition if  $b$  is not in a contracted vertex of  $\overline{G}^-$ , else by P2 in  $\overline{G}^-$ ) and  $P_b$  is the preimage of both  $P(\overline{b}, \overline{G}^-)$  and  $P(\overline{b}, \overline{G}^+)$ .

Finally the last assertion of P2 follows since any  $y \in V(B)$  has  $\overline{y} = \overline{b}$ .  $\diamond$

**Claim 4:** P3 holds in every iteration.

**Proof:** By definition

$$\overline{\zeta}(uv) = \overline{w}(P_{\overline{u}}) + \overline{w}(P_{\overline{v}}) \oplus \overline{w}(uv) = (\overline{w}(P_{\overline{u}}) \oplus \Delta(uv, u)) + (\overline{w}(P_{\overline{v}}) \oplus \Delta(uv, v)) \oplus w(uv)$$

for some  $\oplus \in \{+, -\}$ . Since  $uv$  is an edge of  $\overline{G}$ , P2 shows  $uv \in P_u$  iff  $uv \in P_{\overline{u}}$ . Hence the version of this formula for  $\zeta(uv)$  uses the same sign  $\oplus$  as  $\overline{\zeta}(uv)$ . We evaluate  $\overline{\zeta}(uv)$  using the formula, with Claim 2 providing  $\overline{w}(P_{\overline{u}})$  and the definition of  $\Delta(uv, u)$  giving its value, as follows.

Suppose  $u \notin V(C)$ .  $P_u$  is the preimage of  $P_{\overline{u}}$  by P2. So

$$\overline{w}(P_{\overline{u}}) \oplus \Delta(uv, u) = w(P_u)$$

since  $\overline{w}(P_{\overline{u}}) = w(P_u)$ ,  $\Delta(uv, u) = 0$ .

Suppose  $u \in V(C)$ , say  $u \in V(C)$  for the p-cycle  $C, cc'$ .  $P_c$  is the preimage of  $P_{\overline{u}}$  by P2. Claim 2 shows  $\overline{w}(P_{\overline{u}}) = w(P_c) - 2\omega$ . If  $uv \notin P_u$  then

$$\overline{w}(P_{\overline{u}}) + \Delta(uv, u) = w(P_u)$$

since  $\Delta(uv, u) = 2\omega + w(C_u)$ , and  $P_u = C_u \cup P_c$  by P2 and the optimality of  $P_u$ . If  $uv \in P_u$  then

$$\begin{aligned} \overline{w}(P_{\overline{u}}) - \Delta(uv, u) &= w(P_u), \\ \overline{w}(P_{\overline{u}}) + \Delta(uv, u) &= w(P_u) - 4\omega \end{aligned}$$

since P2 implies  $u = c$  and  $\Delta(uv, u) = -2\omega$ .

The alternatives of P3 all follow, by combining the equations with  $-$  signs for  $uv \in P_u \cap P_v$  and  $+$  signs in all other cases.  $\diamond$  ■

## 7 Background on $f$ -factors

The bulk of this section reviews the approach of [14] based on critical graphs. The review ends by illustrating the shrinking procedure. Then we show that procedure efficiently constructs a generalized shortest-path structure.

To define critical graphs, for each vertex  $v \in V$  define  $f_v$ , the *lower perturbation* of  $f$  at  $v$ , by decreasing  $f(v)$  by 1. Similarly define  $f^v$ , the *upper perturbation* of  $f$  at  $v$ , by increasing  $f(v)$  by 1. Every  $f_v, f^v, v \in V$  is a *perturbation* of  $f$ .  $f \uparrow_v$  stands for a fixed perturbation that is either  $f_v$  or  $f^v$ . A graph is  *$f$ -critical* if it has an  $f'$ -factor for every perturbation  $f'$  of  $f$ .<sup>11</sup>

It is easy to see that a maximum  $f$ -factor of  $G$  can be found by working on the critical graph  $G^+$  formed by adding a vertex  $s$  with edges  $sv, v \in V(G)$  and loop  $vv$ , all of weight 0, and  $f(s) = 1$ : A maximum  $f_s$ -factor is the desired  $f$ -factor.

Recall the linear programming formulation for maximum weight  $f$ -factors, especially the dual problem (e.g. [27, Ch. 32]). We summarize the slight modification used in [14] for critical graphs. Simply put we want the dual variables to be optimum for every perturbation of  $f$ .

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<sup>11</sup>The usual notion of criticality for matching only assumes existence of the factors for lower perturbations. It is easy to see this implies existence for all the upper perturbations. This holds for  $b$ -matching too, but not for general  $f$ -factors [14].

Dual variables for a graph with integral weights are functions  $y : V \rightarrow \mathbb{Z}$  and  $z : 2^V \times 2^E \rightarrow \mathbb{Z}$ . Pairs  $B = (V(B), I(B))$  with nonzero  $z$ -value are called (weighted) “blossoms”.  $z(V)$  is a shorthand for  $z((V, \emptyset))$ . The optimum dual function  $y$  is defined by  $y(v) = -w(F_v)$  [14, Theorem 4.17], as expected from Sec.6.

The  $z$  function has support given by a forest that we now describe. It generalizes the shortest-path structure of Section 6. The usual version corresponds exactly to that structure; it also corresponds to the blossom tree for matching [14]. Our algorithms use a weighted version of this structure – it is the shortest-path structure/blossom tree with every cycle node/blossom of 0  $z$  value contracted into its parent. This *weighted blossom forest* is defined as follows (we give a minor modification of [14, Definition 4.14], using [14, Lemma [14, Lemma 4.12]]). Let  $\mathcal{M}$  be the set of all maximal blossoms of  $G$ .  $\mathcal{M} = \{V\}$  for matching and  $b$ -matching but not generally (for shortest paths  $\mathcal{M}$  is the set of maximal cycle nodes).

(i) Each  $B \in \mathcal{M}$  is the root of a *weighted blossom tree*  $\mathcal{W}$ . Each interior node of  $\mathcal{W}$  is a weighted blossom and each leaf is a vertex of  $G$ . The children of any node  $B$  are the maximal weighted blossoms properly contained in  $B$  plus all vertices of  $G$  contained in  $B$  but no smaller weighted blossom.  $V(B)$  is the set of all leaf descendants of  $B$ , i.e., the vertices in  $B$ . Each vertex of  $G$  belongs to exactly one blossom of  $\mathcal{M}$ .

(ii) The support of  $z$  is  $\{B, V : B \text{ a blossom of a } \mathcal{W}\text{-tree}\}$ .  $z(B) > 0$  for blossoms  $B$  of  $\mathcal{W}$ , while  $z(V)$  may have arbitrary sign.

(iii) Each blossom  $B$  has a set of edges  $I(B) \subseteq \delta(V(B))$ .

(iv) The blossoms of  $\mathcal{M}$  are nodes of a tree  $\mathcal{T}$ . Every edge  $AB$  of  $\mathcal{T}$  belongs to  $I(A) \oplus I(B)$ .

For shortest paths,  $I(B)$  consists of the edge  $e(N)$ . As in shortest paths and matching, a blossom  $B$  covers any edge of  $\gamma(V(B)) \cup I(B)$ . An edge  $uv$  of  $G$  has a value

$$\widehat{yz}(e) = y(e) + z\{B, V : B \text{ a blossom of } \mathcal{W} \text{ that covers } e\}.$$

(Here we use the above convention; recall  $e = \{u, v\}$ .)  $e$  is *underrated* if

$$w(e) \geq \widehat{yz}(e);$$

$e$  is *strictly underrated* if the inequality is strict and *tight* if equality holds. An  $f \uparrow v$ -factor  $F$  respects a blossom  $B$  iff

$$F \cap \delta(V(B)) = \begin{cases} I(B) & v \in V(B) \\ I(B) \oplus e & v \notin V(B), e \text{ is some edge in } \delta(V(B)) \end{cases} \quad (21)$$

(For shortest paths, the first alternative says a shortest  $vt$ -path leaves  $B$  on  $e(B)$  if  $v \in V(B)$ ; otherwise the second line says it contains  $e(B)$  and one other edge ( $t \notin B$ ) or one edge  $\neq e(B)$  ( $t \in B$ )). An  $f \uparrow v$ -factor has maximum weight if it contains every strictly underrated edge, its other edges are tight, and it respects every blossom with positive  $z$ -value. (The optimum  $f$ -factors we use satisfy this criterion.)

As in matching, blossoms are built up from odd cycles, defined as follows [14, Definition 4.3]: An *elementary blossom*  $B$  is a 4-tuple  $(VB, C(B), CH(B), I(B))$ , where  $VB \subseteq V$ ,  $C(B)$  is an odd circuit on  $VB$ ,  $CH(B) \subseteq \gamma(VB) - E(C(B))$ ,  $I(B) \subseteq \delta(VB)$ , and every  $v \in VB$  has  $f(v) = d(v, C(B))/2 + d(v, CH(B) \cup I(B))$ .  $C(B)$ ,  $CH(B)$ , and  $I(B)$  the *circuit*, *chords*, and *incident edges* of  $B$ , respectively. We sometimes use “blossom” or “elementary blossom” to reference the blossom’s

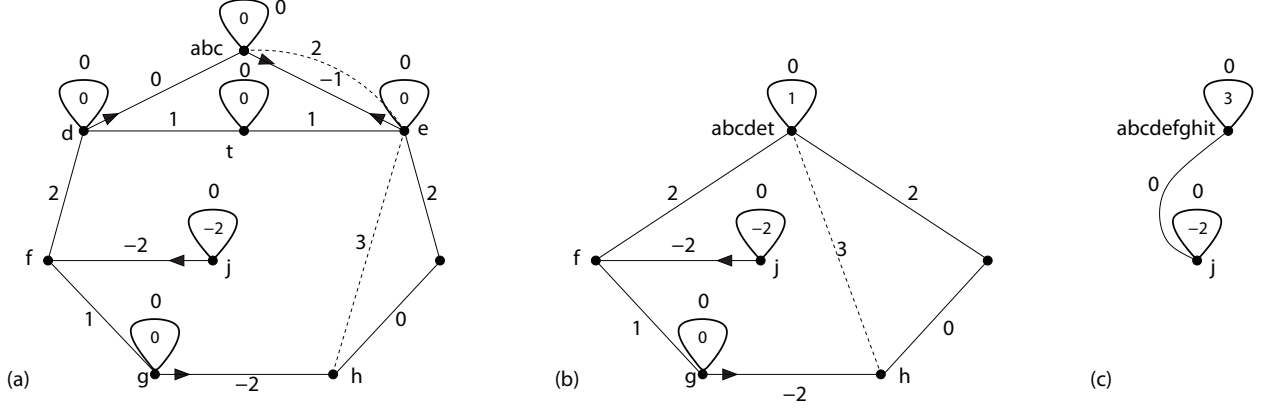


Figure 4: Contracted graphs for nonloop unifactors.

circuit or its odd pair. Blossoms for shortest pairs have  $|I(B)| \leq 1$  (as mentioned) and  $CH(B) = \emptyset$  (this is the import of Lemma 28, especially part (ii) and Claim 2 respectively).

The blossoms for optimum duals are found by repeatedly finding the next blossom and shrinking it. Fig.4 illustrates the shrinking of the 3 cycle nodes of Fig.3, e.g.,  $\{a, b, c\}$  gets shrunk in Fig.4(a). In general the vertices of the blossom are contracted and a loop is added to the contracted vertex. Edges incident to the contracted vertex get their weights adjusted to account for contracted edges. (For shortest paths these edges are the paths  $p(v, N)$ . In Fig.4(a)  $w(be)$  decreases by 1 to account for path  $b, a, c$ ).<sup>12</sup>

For simplicity perturb the edge weights slightly so that no two sets of edges have the same weight. That is, number the edges from 1 to  $m$  and increase the weight of the  $i$ th edge by  $\epsilon^i$  for some  $\epsilon \geq 0$ . For small enough  $\epsilon > 0$ , no two sets of edges have the same weight. Thus any such perturbation has a unique maximum factor which is also maximum for the original weights. Let  $F_v$  and  $F^v$  denote the maximum weight  $f_v$  and  $f^v$ -factors respectively. (Eventually (in (22)) we set  $\epsilon$  to 0 and define the dual function  $z$  using the original weights.) Recall (Sec. 2) the multiset notation  $2S$ ,  $S/2$ ,  $2G$ . Assume that even in  $2G$ , no two sets of edges have the same weight.

We choose the blossom to shrink next using a subgraph that generalizes the  $uv$ -subgraphs of Section 6: Any subgraph of  $2G$  consists of edges of  $G$  at multiplicity 0,1 or 2. A  $2f$ -unifactor is a  $2f$ -factor of  $2G$  whose multiplicity 1 edges form an odd circuit. (When possible we abbreviate “ $2f$ -unifactor” to “unifactor”.) The elementary blossom for a unifactor  $U$  with odd circuit  $C$  is the 4-tuple  $B = (V(C), C, CH(B), I(B))$  where  $CH(B) = (U \cap \gamma(C) - E(C))/2$  and  $I(B) = (U \cap \delta(C))/2$ . For shortest paths note how this defines  $I(B)$  to be  $e(N)$  (the type (b) path to  $b$  ends in multiplicity 2 edge  $e(N)$ ).

The *shrinking procedure* of [14] constructs the blossoms as follows:

Let the next blossom  $B$  be the elementary blossom of the maximum weight proper  $2f$ -unifactor.<sup>13</sup> Shrink  $B$ . Repeat this step until no unifactor exists.

<sup>12</sup>The weight adjustment of [14, Fig.13] also involves a large quantity  $J$ . We omit it since  $J$  is needed only for the proof of [14], not for explaining the result.

<sup>13</sup>“Proper” means (a) the unifactor’s circuit is not a loop created previously when a blossom was contracted; (b) the unifactor respects each previous blossom. Here “respects” is the generalization of (21) to unifactors, e.g., a shortest

Let us describe how this procedure constructs the blossoms and tree of Fig.3. We use Fig.4. The shrinking procedure repeatedly finds the minimum weight proper unifactor. Let  $\zeta^*$  denote its weight.  $\zeta^*$  never decreases from step to step, so we describe the blossoms found at the various values of  $\zeta^*$ . Recall that we treat the shortest-path problem by adding a 0 weight loop at every vertex  $\neq t$ .

$\zeta^* = -2$ : The loops at  $a, b$  and  $j$  are blossoms. (For instance the unifactor for  $j$  corresponds to the shortest  $jt$ -path – it consists of loop  $jj$  (multiplicity 1) and the  $jt$ -path of weight  $-1$  plus a loop at every vertex not on the path (all multiplicity 2). The weight of the unifactor,  $-2$ , is drawn inside the loop at  $j$  in Fig.4(a). Fig.4 does this for all blossoms.)

$\zeta^* = 0$ : The loops at  $d, e, g, t$  are blossoms, as is cycle  $a, b, c$ . Fig.4(a) shows the graph after all these blossoms have been shrunk.

$\zeta^* = 1$ : Contracted vertex  $\{a, b, c\}$  plus vertices  $d, e, t$  form a blossom. Fig.4(b) shows the graph after it has been shrunk.

$\zeta^* = 2$ : The loops at  $f, h$ , and  $i$  are blossoms.

$\zeta^* = 3$ : All vertices but  $j$  form a blossom. Fig.4(c) shows the graph after it is shrunk. The new weight on  $jt$  reflects the contracted edge  $fd$ .

Note that when a loop  $xx$  becomes a blossom, the unifactor's weight is  $2d(x)$ . This corresponds to the bound  $2d(x)$  in Sec.6 mentioned for the Bellman inequality argument. Also, the description for loops in  $\zeta^* = -2$  might seem to imply we need to know the shortest paths to execute this procedure – but see the implementation in Sec.8!

The optimum dual function  $z$  is defined as follows. For any blossom  $B$  let  $U(B)$  be its corresponding unifactor. If  $B$  is not a maximal blossom let  $p(B)$  denote the blossom  $B$  gets contracted into. Let  $w$  be the original (unperturbed) weight function.

$$z(B) = \begin{cases} w(U(B)) & B \text{ a maximal blossom,} \\ w(U(B)) - w(U(p(B))) & B \text{ nonmaximal.} \end{cases} \quad (22)$$

For instance in Fig.3 blossom  $aa$  is created at  $\zeta^* = -2$ ,  $\{a, b, c\}$  is created at  $\zeta^* = 0$ , and  $z(\{a, b, c\}) = -2 - 0 = -2$ .

The following characterization of maximum weight unifactors is central to the analysis:

**Lemma 33** ([14, Lemma 4.5 and Cor. 4.6]). *For any vertex  $v$ , the maximum weight  $2f$ -unifactor containing  $v$  in its circuit is  $F_v + F^v$ . For any edge  $uv$  consider 3 cases:*

*$uv \notin F_u \cup F_v$ : The maximum weight  $2f$ -unifactor containing  $uv$  in its circuit is  $F_u + F_v + uv$ .*

*$uv \in F_u \cap F_v$ : The maximum weight  $2f$ -unifactor containing  $uv$  in its circuit is  $F^u + F^v - uv$ .*

*$uv \in F_v - F_u$ :  $F_u = F^v - uv$ . Furthermore the maximum weight  $2f$ -unifactor containing  $v$  in its circuit is  $F_u + F_v + uv$ , and this unifactor contains  $uv$  as an edge incident to its circuit.*

For shortest paths (i) – (ii) characterize the maximum  $uv$ -subgraph. (iii) corresponds to Lemma 26.

We close this review by reiterating some notation from [14] that we use in the next two sections:  
 $w$ : the given, unperturbed, weight function.

$\mathcal{B}$ : the forest whose nodes are the elementary blossoms found by the shrinking procedure, plus the vertices of  $G$ . The children of a blossom  $B$  are the vertices that get contracted to form  $B$ .

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path that enters and leaves a blossom must contain  $e(N)$ . [14] enforces (b) using the previously mentioned quantity  $J$  that we omit.

$\bar{z}(B)$ : the sum of the dual values  $z(A)$  for every ancestor  $A$  of  $B$  in  $\mathcal{B}$ . If  $B$  does not appear in the weighted blossom forest  $\mathcal{W}$  (defined above) then  $\bar{z}(B) = \bar{z}(A)$  for the blossom  $A$  of  $\mathcal{W}$  that absorbs  $B$  by contraction.

$B_v$ , for any vertex  $v \in V(G)$ : the smallest weighted blossom containing  $v$ .

## 8 Weighted $f$ -factors

This section presents the reduction of maximum weight  $f$ -factors to unweighted  $f$ -factors. As usual we use the shrinking procedure to find the weighted blossoms  $B$  in order of decreasing value  $\bar{z}(B)$ . A blossom is a pair of sets  $(V(B), I(B))$ . We find these pairs in two steps using a graph  $G(\zeta)$ : Having found the blossoms of  $\bar{z}$ -value  $> \zeta$ , we construct  $G(\zeta)$ . Its 2-edge-connected components constitute the  $V$ -sets for all weighted blossoms of  $\bar{z}$ -value  $\zeta$ . The edges of  $I$ -sets that are still unknown are found amongst the bridges of  $G(\zeta)$  or in a related computation.

In our references to [14] we are careful to recall that [14] modifies the given weight function  $w$  in two ways: First, the given edge weights are perturbed so every maximum factor  $f\uparrow_v$  and every maximum  $2f$ -unifactor is unique. Second, in [14] each time a blossom is contracted the weights of its incident edges are modified (by adding large quantities, including a value called  $W$  greater than the sum of all previous edge weights). The reduction of this section has no access to these conceptual modifications, and it must work entirely with the given edge weight function  $w$ .

Our overall strategy is similar to the  $b$ -matching algorithm of Section 5: We assemble the desired maximum weight  $f\uparrow_v$ -factor from its subgraphs that lie in the various nodes of the weighted blossom tree. The details are similar to Section 5, and it is implicit in [14] that these details work. But for completeness the next lemma proves the necessary properties.

Any blossom  $B$  in a  $\mathcal{W}$ -tree has an associated graph  $\overline{G}(B)$ . Its vertices are the children of  $B$ , with every blossom child contracted. Its edges are the underrated edges that join any two of its vertices (this includes underrated loops). Informally the lemma states that the edges of any maximum  $f\uparrow_v$ -perturbation in  $\overline{G}(B)$  satisfy all relevant constraints.

**Lemma 34.** *For any  $v \in V(G)$ , any perturbation  $f\uparrow_v$ , and any  $B$  in  $\mathcal{W}$ , let  $\overline{B} \subseteq \delta(B)$  be a set of edges that respects  $B$ . If  $\overline{B}$  has the form  $I \oplus e$  assume  $e$  is tight. Then  $\overline{G}(B) \cup \overline{B}$  has a subgraph  $F$  containing  $\overline{B}$  wherein*

- (a) *every child of  $B$  that is a vertex  $x \in V(G)$  has  $f\uparrow_v(x) = d(x, F)$ ;*
- (b)  *$F$  respects every child of  $B$  that is a contracted blossom.*

*Proof.* The two possible forms for  $\overline{B}$  are  $I(B)$  and  $I(B) \oplus e$ . We first show that  $I(B)$  is unique and the lemma holds when  $\overline{B} = I(B)$ .

For any  $v \in V(B)$ , any maximum  $f\uparrow_v$ -factor  $F$  respects  $B$ , i.e., it has  $F \cap \delta(B) = I(B)$ . (This follows from the optimality of the duals.) So  $I(B)$  is unique. Also this factor  $F$  proves the lemma when  $\overline{B} = I(B)$ , i.e.,  $F$  satisfies (a) and (b).

For  $v \notin V(B)$  any set respecting  $B$  has the form  $I(B) \oplus e$ ,  $e \in \delta(B)$ . By assumption  $e$  is tight. Let  $e = xy$  with  $y \in V(B)$ . If  $e \in I(B)$  then  $F = F^y - xy$  is a maximum  $f_x$ -perturbation, with  $F \cap \delta(B) = I - e$ . (This follows from the optimality criterion of the duals.) Similarly if  $e \in \delta(B) - I(B)$  then  $F = F_y + xy$  is a maximum  $f^x$ -perturbation with  $F \cap \delta(B) = I + e$ . As before, the maximum factor  $F$  proves the lemma. ■

## 8.1 Finding the $V(B)$ -sets

The reduction is given the quantities,  $w(F_v), w(F^v), v \in V(G)$ . So it can use these quantities:

For each vertex  $v \in V$ ,  $\zeta_v = w(F_v) + w(F^v)$ .

For each edge  $uv \in E$ ,  $\zeta_{uv} = w(F_u) + w(F_v) + w(uv)$  and  $\zeta^{uv} = w(F^u) + w(F^v) - w(uv)$ .

For a vertex  $v$  let  $UNI(v)$  be the maximum  $2f$ -unifactor containing  $v$  in its circuit. Lemma 33 shows  $\zeta_v = w(UNI(v))$ . Recalling the definition of dual variables (22) we get

$$\zeta_v = \bar{z}(B_v).$$

For any edge  $uv$  of  $G$  let  $UNI(uv)$  be the maximum  $2f$ -unifactor containing  $uv$  in its circuit. We classify  $uv$  as type 0, 1 or 2 depending on the number of sets  $F_u, F_v$  that contain  $uv$ . Specifically  $uv \in E$  is

type 0 if  $uv \notin F_u \cup F_v$ ;

type 2 if  $uv \in F_u \cap F_v$ ;

type 1 if  $uv \in F_u \oplus F_v$ . Additionally a type 1 edge  $uv$  is type  $1_\zeta$  if  $\zeta_u, \zeta_v > \zeta$ .

Of course the type of an edge is unknown to the reduction! Lemma 33 extends to give the following combinatoric interpretations of  $\zeta_{uv}$  and  $\zeta^{uv}$ . In contrast to  $\zeta_v$  these interpretations are also unknown to the reduction.

**Lemma 35.** *Consider any edge  $uv$  of  $G$ .*

(i) *If  $uv$  is type 0 then  $UNI(uv) = F_u + F_v + uv$ . Thus  $w(UNI(uv)) = \zeta_{uv}$ . Furthermore  $\zeta_{uv} \leq \zeta_u, \zeta_v \leq \zeta^{uv}$ .*

(ii) *If  $uv$  is type 2 then  $UNI(uv) = F^u + F^v - uv$ . Thus  $w(UNI(uv)) = \zeta^{uv}$ . Furthermore  $\zeta^{uv} \leq \zeta_u, \zeta_v \leq \zeta_{uv}$ .*

(iii) *Suppose  $uv$  is type 1 with  $uv \in F_v - F_u$ . Then  $UNI(v) = F_u + F_v + uv$  and  $uv$  is incident to the circuit of  $UNI(v)$ . So  $w(UNI(v)) = \zeta_v = \zeta_{uv}$ . Furthermore  $\zeta^{uv} = \zeta_u$ .*

*Proof.* Lemma 33 gives the characterization of the various unifactors. The relations between the various  $\zeta$  quantities all follow easily from this simple identity: Any edge  $uv \in E$  satisfies

$$\zeta_{uv} + \zeta^{uv} = \zeta_u + \zeta_v. \quad (23)$$

For instance to prove part (i), the relation  $w(U_{uv}) \leq w(U_u), w(U_v)$  translates to  $\zeta_{uv} \leq \zeta_u, \zeta_v$ . Now (23) implies  $\zeta_u, \zeta_v \leq \zeta^{uv}$ . ■

We define the graph  $G(\zeta)$ , for any real value  $\zeta$ : Its vertices are the vertices of  $G$  with all blossoms of  $\bar{z}$ -value  $> \zeta$  contracted. Its edge set is

$$E(G(\zeta)) = \{uv : uv \in E(G), \min\{\zeta_{uv}, \zeta^{uv}\} \geq \zeta\}.$$

The reduction can construct  $G(\zeta)$ , since previous iterations have identified the blossoms of  $\bar{z}$ -value  $> \zeta$ .

Recall a blossom  $B$  of  $\mathcal{B}$  is constructed as an elementary blossom in a contracted graph  $\bar{G}$ . As such it has a circuit  $C(B)$ .  $C(B)$  consists of edges that are images of edges of  $G$ , as well as blossom loops (resulting from contractions). In the lemma statement below,  $E(C(B)) \cap E(G)$  denotes the edges of  $G$  whose images belong to  $C(B)$ .

**Lemma 36.** *For any blossom  $B$  of  $\mathcal{B}$ ,  $E(C(B)) \cap E(G) \subseteq E(G(\bar{z}(B)))$ .*

*Proof.* Consider any edge  $uv \in C(B)$ . The desired relation  $uv \in E(G(\bar{z}(B)))$  amounts to the inequality  $\min\{\zeta_{uv}, \zeta^{uv}\} \geq \bar{z}(B)$ . (Note that  $uv$  cannot be a loop in  $G(\bar{z}(B))$  since  $uv$  joins distinct blossoms of  $C(B)$ .) We will prove the desired inequality using the fact that every edge  $uv \in C(B)$  is tight, proved in [14, Theorem 4.17].<sup>14</sup> Consider three cases:

**Case  $uv$  is type 0:** Tightness means  $y(u) + y(v) + \bar{z}(B) = w(uv)$ . Equivalently  $\bar{z}(B) = w(F_u) + w(F_v) + w(uv)$ . Thus  $\bar{z}(B) = \zeta_{uv}$  and Lemma 35(i) shows  $uv \in G(\bar{z}(B))$ .

**Case  $uv$  is type 2:** Tightness means  $y(u) + y(v) + \bar{z}(B_u) + \bar{z}(B_v) - \bar{z}(B) = w(uv)$ . As noted above

$$w(F_v) + w(F^v) = \zeta_v = \bar{z}(B_v)$$

and similarly for  $u$ . Substituting this relation gives  $w(F^u) + w(F^v) - w(uv) = \bar{z}(B)$ . Thus  $\bar{z}(B) = \zeta^{uv}$  and Lemma 35(ii) shows  $uv \in G(\bar{z}(B))$ .

**Case  $uv$  is type 1:** Wlog  $uv \in F_v - F_u$ . Tightness means  $y(u) + y(v) + \bar{z}(B_v) = w(uv)$ . Equivalently

$$\bar{z}(B_v) = w(F_u) + w(F_v) + w(uv).$$

With  $\bar{z}(B_v) \geq \bar{z}(B)$  (since  $B$  is an ancestor of  $B_v$ ) this gives  $\zeta_{uv} \geq \bar{z}(B)$ .

The (last) displayed equation is equivalent to  $w(F_v) + w(F^v) = w(F_u) + w(F_v) + w(uv)$ . Rearranging gives  $w(F^u) + w(F^v) - w(uv) = w(F^u) + w(F_u) = \bar{z}(B_u) \geq \bar{z}(B)$ , i.e.,  $\zeta^{uv} \geq \bar{z}(B)$ . ■

In  $G(\zeta)$  every type 0 edge is in a blossom circuit (Lemma 35(i), which refers to the given graph, and [14, Lemma 4.13], which shows the relation of Lemma 35(i) is preserved as blossoms are contracted). The same holds for every type 2 edge. (However note that an arbitrary type 0 or 2 edge needn't belong to a blossom circuit – it may not appear in any  $G(\zeta)$  graph because of blossom contractions.)

In contrast a type 1 edge of  $G(\zeta)$  may or may not be in a blossom circuit. The  $1_\zeta$  edges obey the following generalization of Corollary 27 for shortest paths.

**Lemma 37.** *In any graph  $G(\zeta)$  the  $1_\zeta$  edges are acyclic.*

*Proof.* We start with a relation between the set  $I(A)$  of a blossom  $A \in \mathcal{B}$  and the same set when  $A$  is contracted, i.e., set  $I(a)$  for blossom vertex  $a$ . In  $G(\zeta)$  suppose  $a$  is a blossom vertex and vertex  $b \neq a$ . Recall that  $F_a \cap \delta(a) = I(a)$  and  $F_b$  respects blossom  $a$  ([14, Corollary 4.11], which says that as expected, every maximum perturbation  $F \uparrow_v$  respects every maximum proper unifactor's blossom). Thus

$$|(F_a \cap \delta(a)) \oplus (F_b \cap \delta(a))| = |I(a) \oplus (F_b \cap \delta(a))| = 1. \quad (24)$$

We use (24) to prove the following:

**Claim** *In  $G(\zeta)$  consider a blossom vertex  $c$  and distinct edges  $e, f \in \delta(c)$ . Suppose vertices  $b, c, c'$  have  $e \in F_b \oplus F_c$  and  $f \in F_c \oplus F_{c'}$ . Then  $f \in F_b \oplus F_{c'}$ .*

**Proof:** We can assume  $b \neq c$  since otherwise the lemma is tautologous. Since  $F_b$  respects blossom  $c$  and  $e, f \in \delta(c)$ ,  $e \in F_b \oplus F_c$  implies  $f \notin F_b \oplus F_c$  (by (24)). Combining with the hypothesis

<sup>14</sup>In matching and  $b$ -matching tightness is forced by the fact that every edge of a blossom circuit is in some maximum perturbation. The analogous statement fails for  $f$ -factors – an “exceptional” circuit edge may belong to no maximum perturbation at all or to every maximum perturbation [14].

$f \in F_c \oplus F_{c'}$  gives  $f \in (F_b \oplus F_c) \oplus (F_c \oplus F_{c'}) = F_b \oplus F_{c'}$ .  $\diamond$

Now consider a cycle of  $1_\zeta$  edges in  $G(\zeta)$ , say  $a, b, \dots, y, z$  with  $z = a$ . Type 1 means  $ab \in F_a \oplus F_b$ . This immediately shows the cycle has  $\geq 2$  edges. We will show edge  $yz = ya$  also belongs to  $F_a \oplus F_b$ .  $ya$  may be parallel to  $ab$  (a length 2 cycle) but  $ya$  is not the same edge as  $ab$  (i.e., we do not have a length 1 cycle, since a loop  $aa$  is not type 1). So we get 2 distinct edges in  $\delta(a) \cap (F_a \oplus F_b)$ . This contradicts (24).

It is convenient to also denote the cycle as  $a, c^0 = b, c^1, \dots, c^{r-1} = y, c^r = z$ . Type 1 means  $c^{i-1}c^i \in F_{c^{i-1}} \oplus F_{c^i}$ . Inductively assume  $c^{i-1}c^i \in F_b \oplus F_{c^i}$ . (This holds for  $i = 1$ .) Since  $c^i c^{i+1} \in F_{c^i} \oplus F_{c^{i+1}}$  the claim (with  $c = c^i$ ,  $c' = c^{i+1}$ ) shows  $c^i c^{i+1} \in F_b \oplus F_{c^{i+1}}$ . Thus induction shows  $c^{r-1}c^r \in F_b \oplus F_{c^r}$ , i.e.,  $yz \in F_b \oplus F_z$ , as desired.  $\blacksquare$

The reduction processes the graphs  $G(\zeta)$  for  $\zeta$  taking on the distinct values in

$$\Omega = \{\min\{\zeta_{uv}, \zeta^{uv}\} : uv \text{ an edge of } G\}$$

in descending order. For any  $\zeta \in \Omega$  let  $\zeta^-$  be any value strictly between  $\zeta$  and the next largest value in  $\Omega$ . Observe that  $G(\zeta^-)$  is the graph  $G(\zeta)$  with every blossom of  $\bar{z}$ -value  $\geq \zeta$  contracted.

**Corollary 38.** *For any  $\zeta \in \Omega$ ,  $G(\zeta^-)$  is a forest.*

*Proof.* In  $G(\zeta)$  every type 0 or 2 edge is in a blossom circuit (as indicated after Lemma 36). So  $G(\zeta^-)$ , which has all these blossoms contracted, has only type 1 edges  $uv$ . Furthermore since  $\min\{\zeta_{uv}, \zeta^{uv}\} \geq \zeta$ , Lemma 35(iii) shows  $\zeta_u, \zeta_v \geq \zeta$ . In other words  $uv$  is type  $1_{\zeta^-}$ . Thus Lemma 37 shows  $G(\zeta^-)$  is acyclic.  $\blacksquare$

The next lemma shows how the reduction finds the vertex sets of the blossoms of  $\mathcal{W}$  that have  $\bar{z}$ -value  $\zeta$ .

**Lemma 39.** *As vertex sets, the 2-edge-connected components of  $G(\zeta)$  are precisely the weighted blossoms of  $\bar{z}$ -value  $\zeta$ .*

*Proof.* We first show that each blossom of  $\mathcal{B}$  is 2-edge-connected. More precisely let  $B$  be a node of  $\mathcal{B}$  with  $\bar{z}(B) = \zeta$ .

**Claim** *In  $G(\zeta)$  the subgraph of edges  $\bigcup\{C(A) : \text{node } A \text{ of } \mathcal{B} \text{ descends from } B \text{ and } \bar{z}(A) = \zeta\}$  is 2-edge-connected.*

**Proof:** The argument is by induction on the number of descendants  $A$ . Recall  $\mathcal{B}$  is constructed by repeatedly finding the next elementary blossom  $B$  and contracting it.

When  $B$  is found, each vertex of its circuit is either (i) an original vertex of  $G$ , or (ii) a contracted blossom of  $\bar{z}$ -value  $> \zeta$ , or (iii) a contracted blossom of  $\bar{z}$ -value  $\zeta$ . (Recall the definition of  $z$ , (22). The type (iii) blossoms are blossoms with  $z$ -value 0.) Vertices of type (i) or (ii) are vertices of  $G(\zeta)$ . Vertices of type (iii) have 2-edge-connected subgraphs in  $G(\zeta)$  by induction. Each original edge of  $C(B)$  is contained in  $G(\zeta)$  (Lemma 36). Since  $C(B)$  is a circuit when  $B$  is formed, it completes a 2-edge-connected subgraph of  $G(\zeta)$ . This completes the induction.  $\diamond$

Now starting with  $G(\zeta)$ , contract each of the above 2-edge-connected subgraphs that corresponds to a maximal blossom of  $\bar{z}$ -value  $\zeta$ . We get the acyclic graph  $G(\zeta^-)$  (Corollary 38). So the contracted subgraphs are precisely the 2-edge-connected components of  $G(\zeta)$ .  $\blacksquare$

In summary we find all the  $V(B)$ -sets as follows.

Compute all values  $\zeta_{uv}, \zeta^{uv}$ ,  $uv$  an edge of  $G$ . Then repeat the following step for  $\zeta$  taking on the distinct values in  $\Omega$  in decreasing order:

Construct  $G(\zeta)$ , contracting all  $V(B)$ -sets of blossoms of  $\bar{z}$ -value  $> \zeta$ . Find the 2-edge-connected components of  $G(\zeta)$  and output them as the  $V(B)$ -sets of blossoms of  $\bar{z}$ -value  $\zeta$ .

In addition, output the graph  $G(\zeta^-)$  for the final value of  $\zeta$ .

It is easy to modify the output to get most of the weighted blossom forest  $((i)-(iv)$  of Sec.7) for  $z$ :

(i) The weighted blossom trees  $\mathcal{W}$  are constructed from the containment relation for the 2-edge-connected components.

(ii)  $z(V)$  is the final value of  $\zeta$ . Consider a weighted blossom  $B \neq V$ . Let it be formed in the graph  $G(\zeta)$ . Then  $z(B) = \zeta - \zeta'$ , where  $G(\zeta')$  is the graph in which the parent of  $B$  (in  $\mathcal{W}$ ) is formed, or if  $B \in \mathcal{M} - V$ ,  $\zeta' = z(V)$ . Clearly  $z(B) > 0$ .

(iii) The  $I(B)$ -sets are computed in the next section.

(iv) The tree  $\mathcal{T}$  is the final graph  $G(\zeta^-)$ . In proof  $G(\zeta^-)$  is a forest (Corollary 38). Its vertices are the contractions of the blossoms of  $\mathcal{M}$ , since every vertex in a critical graph is in a blossom (Lemma 33).  $G(\zeta^-)$  is a tree since a critical graph is connected. Finally every edge  $AB$  of  $G(\zeta^-)$  belongs to  $I(A) \oplus I(B)$  since it is type 1 (and any blossom  $C$  has  $I(C) = F_C \cap \delta(C)$ ).

We conclude the section by estimating the time for this procedure. The values in  $\Omega$  are sorted into decreasing order in time  $O(m \log n)$ . We use a set-merging algorithm to keep track of two partitions of  $V(G)$ , namely the connected components of the  $G(\zeta)$  graphs, and the 2-edge connected components. The total time for set merging is  $O(m + n^2)$ .

The total time for the rest of the procedure is  $O(m + n^2)$ . In proof, there are  $\leq m$  iterations ( $|\Omega| \leq m$ ). An iteration that does not change either partition (because its new edges are contracted) uses  $O(1)$  time for each new edge. There are  $\leq 2n$  other iterations (since each of these iterations decreases the number of connected components or 2-edge connected components). Each such iteration uses linear time, i.e.,  $O(1)$  time per vertex or edge of  $G(\zeta)$ . We complete the proof by showing that  $< n$  edges belong to  $> 1$  graph  $G(\zeta)$ .

A type 0 or 2 edge is in one graph  $G(\zeta)$ . A type 1 edge first appears in  $G(\zeta)$  for  $\zeta = \min\{\zeta_u, \zeta_v\}$ . Either it gets contracted in this iteration (and so is in just one  $G(\zeta)$  graph) or it is a  $1_\zeta$  edge in any future  $G(\zeta)$  graph that contains it. Any  $G(\zeta)$  has  $< n$   $1_\zeta$  edges (Lemma 37).

## 8.2 Finding the $I(B)$ -sets

A  $\zeta$ -blossom is a blossom of  $\mathcal{W}$  with  $\bar{z}$ -value  $\zeta$ . The iteration of the reduction for  $\zeta$  finds  $I(B)$  for each  $\zeta$ -blossom  $B$ . For any  $y \in V(B)$  define the set

$$I(y) = I(B) \cap \delta(y).$$

So any blossom  $B$  has  $I(B) = \bigcup \{I(y) : y \in V(B)\}$  (this is part of the definition of blossoms [14, Definition 4.14]; it also easily follows from (iv) of the definition of the weighted blossom tree, since every maximum  $F \uparrow v$  respects every blossom). The iteration for  $\zeta$  computes the sets  $I(y)$  for all

vertices  $y$  with  $\zeta_y = \zeta$ . Clearly we can combine these sets with sets  $I(B')$  known from previous iterations (for values  $\zeta' > \zeta$ ) to find the  $I$ -set of each  $\zeta$ -blossom.

We compute the  $I(y)$ -sets in two steps. For  $y$  and  $B$  as above (i.e.,  $B$  a  $\zeta$ -blossom,  $y \in V(B)$ ,  $\zeta_y = \zeta$ ) define the set

$$I_0(y) = \{xy : xy \in \delta(V(B)), \zeta_y > \zeta^{xy} \text{ or } \zeta_y = \zeta_{xy} \neq \zeta_x\}.$$

**Lemma 40.**  $I_0(y) \subseteq I(y)$ .

*Proof.* Consider an edge  $xy \in I_0(y)$ . Since  $xy \in \delta(V(B))$  we have  $xy \in I(y)$  iff  $xy \in F_y$  ( $F_y$  respects  $B$ ). The latter certainly holds if  $xy$  is type 2. So it suffices to show  $xy$  is not type 0, and  $xy \in F_y$  if  $xy$  is type 1.

Suppose  $xy$  is type 0. Lemma 35(i) shows  $\zeta_y \leq \zeta^{xy}$ , i.e., the first alternative in the set-former for  $I_0(y)$  does not hold. If the second alternative holds we have  $\zeta = \zeta_y = \zeta_{xy}$ , so Lemma 35(i) implies  $x \in V(B)$ , contradicting  $xy \in \delta(V(B))$ .

Suppose  $xy$  is type 1. The first alternative in the set-former for  $I_0(y)$  implies  $\zeta_y \neq \zeta^{xy}$  and the second alternative has  $\zeta_x \neq \zeta_{xy}$ . Both relations imply  $\zeta_y = \zeta_{xy}$  and  $xy \in F_y$  (Lemma 35(iii)). ■

To find the remaining  $I$ -edges for  $G(\zeta)$ , define the set

$$IE = \{uv : uv \text{ joins distinct } \zeta\text{-blossoms, } \zeta = \zeta_u = \zeta_v = \zeta_{uv}\}.$$

**Lemma 41.** Any vertex  $y$  with  $\zeta_y = \zeta$  has  $I(y) - I_0(y) \subseteq IE$ .

*Proof.* Let  $B$  be the  $\zeta$ -blossom containing  $y$ . Suppose  $xy \in I(y)$ . Thus  $xy \in F_y$ , making  $xy$  type 1 or 2. If type 2, Lemma 35(ii) shows  $\zeta^{xy} \leq \zeta_y$ . Furthermore with  $\zeta_y = \zeta$  and  $x \notin V(B)$  it shows the inequality is strict. Thus  $xy \in I_0(y)$ . If type 1, Lemma 35(iii) shows  $\zeta_y = \zeta_{xy}$ . If  $\zeta_x \neq \zeta_y$  then  $xy \in I_0(y)$ . If  $\zeta_x = \zeta_y$  then  $xy \in IE$ , since  $xy \in \delta(V(B))$  shows  $x$  and  $y$  are in different  $\zeta$ -blossoms. ■

$IE$  is a subgraph of  $G(\zeta^-)$  ( $uv \in IE$  has  $\zeta^{uv} = \zeta_{uv} = \zeta$  by (23)). Now observe that every nonisolated vertex of  $G(\zeta^-)$  is a contracted blossom. (In proof, any edge  $xy$  of  $G(\zeta^-)$  belongs to  $G(\zeta)$ , so  $\min\{\zeta_{xy}, \zeta^{xy}\} \geq \zeta$ . Furthermore  $xy$  is type 1, so Lemma 35(iii) shows  $\min\{\zeta_x, \zeta_y\} = \min\{\zeta_{xy}, \zeta^{xy}\}$ . Thus  $\min\{\zeta_x, \zeta_y\} \geq \zeta$ .) We conclude that every edge  $xy$  of  $G(\zeta^-)$  has  $xy \in I(x) \oplus I(y)$  (since every edge of  $G(\zeta^-)$  is type 1).

So to complete the computation of the  $I(B)$ -sets we need only decide which alternative ( $xy \in I(x)$  or  $xy \in I(y)$ ) holds for each edge  $xy$  of  $G(\zeta^-)$ . (Note that an edge  $xy$  of  $G(\zeta^-)$  needn't be in  $IE$  – the blossoms containing  $x$  and  $y$  may not be  $\zeta$ -blossoms. But this causes no harm.) We accomplish this classification using the acyclicity of  $G(\zeta^-)$ , as follows.

Let  $T$  be a nontrivial tree of  $G(\zeta^-)$ . Let  $B$  be a leaf of  $T$ , incident to edge  $xy$  of  $T$  with  $y \in B$ . Since  $F_y$  respects  $B$ ,  $f_y(B) + |I(B)|$  is even (recall the definition of respect). Thus

$$(*) \quad xy \in I(B) \text{ iff } f_y(B) + |I(B) - xy| \text{ is odd.}$$

All edges of  $I(B) - xy$  are known. This follows from Lemma 41 if  $\zeta = \zeta_y$  (recall  $I(B) - IE$  is known). If  $\zeta < \zeta_y$  it holds since all of  $I(B)$  is actually known. So  $(*)$  can be used to add  $xy$  to exactly one of the sets  $I(x)$  or  $I(y)$ . Thus the following algorithm correctly classifies each edge of  $G(\zeta^-)$ .

In the (current) forest  $G(\zeta^-)$ , let  $B$  be a leaf of a nontrivial tree  $T$ . Halt if no such  $T$  exists. Let  $xy$ ,  $y \in B$ , be the unique edge of  $T$  incident to  $B$ . Assign  $xy$  to exactly one of  $I(x)$  or  $I(y)$ , using  $(*)$ . Then delete  $B$  from  $T$  and repeat.

The total time to compute  $I(B)$ -sets is  $O(m + n^2)$ . Specifically, a set  $I_0(y)$  is computed in the iteration where  $\zeta = \zeta_y$ , using  $O(1)$  time on each edge incident to  $y$ . The algorithm for processing  $IE$  is executed in each iteration where some  $y$  has  $\zeta_y = \zeta$ . Each such execution uses  $O(n)$  time, giving time  $O(n^2)$  in total.

### 8.3 Finding a maximum factor

Assume we are given the dual functions  $y$ , and  $z$  in the form of its weighted blossom forest. We show how to find a maximum lower or upper perturbation  $F \updownarrow_v$  for any given  $v \in V$ , in total time  $O(\phi^\omega)$ , with high probability. Let  $F$  be the desired maximum perturbation ( $F = F_v$  or  $F = F^v$ ).

The procedure is in three steps. It halts with the set  $F_0$  equal to  $F$ . Initially  $F_0$  is empty and we add edges to  $F_0$  as they become known.

#### Edges of $\mathcal{T}$

The first step determines  $F \cap \mathcal{T}$  and adds these edges to  $F_0$ . Consider any edge  $e$  of  $\mathcal{T}$ . Let  $X$  and  $X'$  be the sets of the partition of  $V(G)$  induced by the connected components of  $\mathcal{T} - e$ . Since  $F$  is an  $f \updownarrow_v$ -factor,  $f_v(X)$  counts every edge of  $F \cap \gamma(X)$  twice. So

$$e \in F \text{ iff } f_v(X) \text{ is odd.}$$

Clearly we can implement this test to find all edges of  $F \cap \mathcal{T}$  in time  $O(n^2)$ .

#### Strictly underrated edges

The second step calculates  $\widehat{yz}(e)$  for each edge  $e \in E(G)$ . If  $e$  is strictly underrated it is added to  $F_0$ .

To do the calculation efficiently assume each blossom is labelled with its  $\bar{z}$ -value. Let  $e = uv$ . Let  $B$  be the nearest common ancestor of  $u$  and  $v$  in the blossom tree containing  $u$  and  $v$ . Then

$$\widehat{yz}(uv) = y(u) + y(v) + \begin{cases} \bar{z}(B) & uv \notin I(B_u) \cup I(B_v) \\ \bar{z}(B_u) & uv \in I(B_u) - I(B_v) \\ \bar{z}(B_u) + \bar{z}(B_v) - \bar{z}(B) & uv \in I(B_u) \cap I(B_v). \end{cases}$$

This step uses total time  $O(m)$ . (Nearest common ancestors are found in  $O(1)$  time.)

It is convenient to ignore these strictly underrated edges in the rest of the discussion. So assume the degree-constraint function  $f \updownarrow_v$  has been decreased to account for the strictly underrated edges, i.e., from now on  $f \updownarrow_v$  denotes the residual degree constraint.

#### Edges of blossoms

We turn to the third step of the procedure. As in shortest paths and  $b$ -matching, we process the weighted blossoms in a top-down fashion. Consider a weighted blossom  $B$ . Assume the set  $F \cap \delta(B)$  is known. The first step ensures this for a root  $B \in \mathcal{M}$ . We will find the edges of  $F$  that belong to  $\overline{G}(B)$  and add them to  $F_0$ . Note that these edges complete the sets  $F \cap \delta(A)$ ,  $A$  a child of  $B$ . So if

$A$  is a blossom we can process it the same way. Thus we can process every weighted blossom this way.

Form a graph  $H$  as  $\overline{G}(B)$  with the strictly underrated edges deleted. Define  $e \in E(G) \cup \{\emptyset\}$  by the relation

$$F \cap \delta(B) = I(B) \oplus e.$$

If  $v \notin B$  this defines  $e$  as an edge, and if  $v \in B$  it defines  $e$  as  $\emptyset$  (recall  $F$  respects  $B$ ).

Note that if  $e$  is an edge it is tight. In proof, take any  $y \in V(B)$ . Then  $(F \cap \delta(B)) \oplus (F_y \cap \delta(B)) = (I(B) \oplus e) \oplus I(B) = e$ . So  $e \in F \oplus F_y$  implies  $e$  is underrated but not strictly underrated, whence  $e$  is tight.

The desired subgraph of  $H$  is specified in Lemma 34 and we find it as follows. Let  $f'$  be the degree constraint function for  $H$ . Let  $x$  be a vertex of  $H$ . If  $x$  is a vertex of  $G$  then it has degree constraint

$$f'(x) = f \uparrow_v(x) - |\delta(x, F \cap \delta(B))|.$$

If  $x$  is a contracted blossom  $A$  then the edges of  $F$  that are incident to  $x$  are governed by the fact that  $F$  respects  $A$ . Specifically  $F \cap \delta(A)$  is  $I(A)$  if  $v \in A$ , and  $I(A) \oplus g$  for some edge  $g \in \delta(A)$  if  $v \notin A$ .

[14, Section 4.4] models all the above constraints on the desired subgraph so that it corresponds to a maximum weight  $f$ -factor on  $H$ . It does this by redefining the edge weights (in fact it uses weights that are much larger than the given ones – see [14, Fig.13]). This is inappropriate for the current context, since we wish to find the desired subgraph using a routine for unweighted  $f$ -factors.

We model the constraint for  $A$  using the blossom substitute of Fig.5. As indicated, the new vertices  $a, c$  and each  $a_k$  all have degree constraint 1, and

$$f'(b) = \begin{cases} 0 & v \in A \text{ or } e \text{ an edge in } \delta(A) \\ 1 & \text{otherwise.} \end{cases}$$

The following claim shows this substitute faithfully models the constraints on  $A$ . Let  $S$  be a set of edges in the substitute. Let the images of these edges in  $H$  be the set  $SB$  (e.g.,  $i_k a_k$  in  $S$  corresponds to  $i_k A$  in  $SB$ ). Let

$$S\overline{B} = F \cap \delta(B) \cap \delta(A) = (I(B) \oplus e) \cap \delta(A).$$

**Claim**  $S$  satisfies the degree constraints of the blossom substitute for  $A$  iff  $SB \cup S\overline{B}$  respects  $A$ .

**Proof:** Consider the two possible values of  $f'(b)$ .

**Case**  $f'(b) = 0$ :  $S$  satisfies the degree constraints iff it consists of edge  $ac$  and every edge  $a_k i_k$  (but no edge  $b j_k$ ) i.e.,  $SB = I(A) \cap \gamma(B)$ . This is equivalent to

$$SB \cup S\overline{B} = (I(A) \oplus e) \cap \delta(A).$$

If  $v \in A$  then  $e = \emptyset$ , and the displayed equation becomes  $SB \cup S\overline{B} = I(A)$  which is equivalent to  $SB \cup S\overline{B}$  respecting  $A$ . Similarly if  $v \notin A$  and  $e \in \delta(A)$  the displayed equation becomes  $SB \cup S\overline{B} = I(A) \oplus e$  which is equivalent to  $SB \cup S\overline{B}$  respecting  $A$ .

**Case**  $f'(b) = 1$ ,  $v \notin A$ , and  $e \notin \delta(A)$ : The assumption  $e \notin \delta(A)$  implies  $S\overline{B} = I(A) \cap \delta(B)$ .

The degree constraint  $f'(c) = 1$  means  $S$  contains either edge  $ac$  or  $bc$ . The former makes  $S$  contain each edge  $a_k i_k$  and one edge  $b j_h$ , so  $SB \cup S\overline{B} = I(A) + A j_h$ . The latter makes  $S$  contain one edge  $aa_h$  and the edges  $a_k i_k, k \neq h$ , so  $SB \cup S\overline{B} = I(A) - A i_h$ . The assumption  $v \notin A$  in this

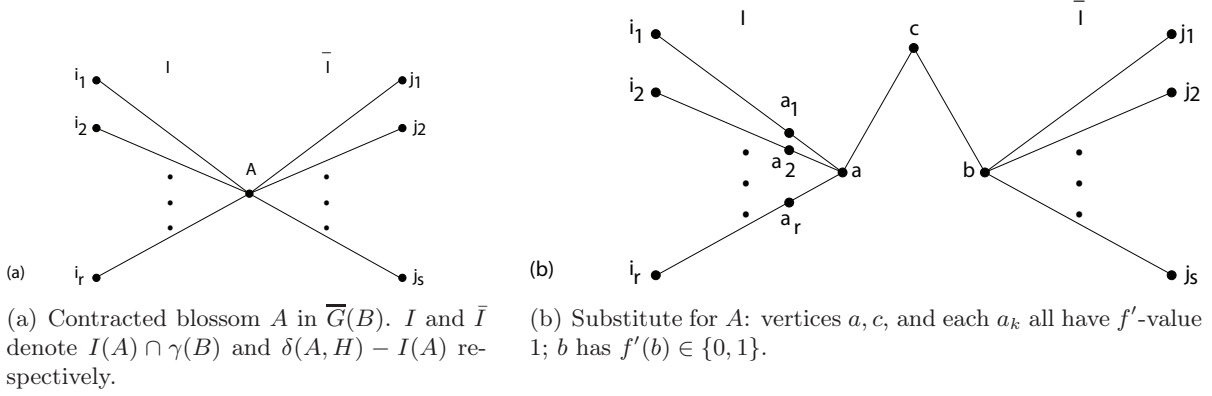


Figure 5: Blossom substitute.

case shows that the two possibilities combined are equivalent to  $SB \cup S\bar{B}$  respecting  $A$ .  $\diamond$

In summary the algorithm for the third step forms the above variant of  $H$  (note the tight edges have been identified in the second step). It finds an  $f'$ -factor on  $H$  and adds these edges to  $F_0$ .

Lemma 34 shows that the desired factor on  $H$  exists. (Note the hypothesis is satisfied, i.e.,  $e$  is tight.) Now it is easy to see that the totality of edges added to  $F_0$  (by all three steps) achieves the optimality condition given in Section 7 for the desired maximum weight  $f \uparrow_v$ -factor.

The time for the third step is  $O(\phi^\omega)$ . To prove this it suffices to show that the total number of vertices in all  $H$  graphs is  $O(\phi)$ . There are  $O(n)$  weighted blossoms, and hence  $O(n) = O(\phi)$  vertices of type  $a, b$  or  $c$  in blossom substitutes. An edge  $e$  in an  $I(B)$ -set occurs in only one  $H$  graph – the graph corresponding to  $\overline{G}(p(B))$ , for  $B$  the maximal set with  $e \in I(B)$  and  $p(B)$  the parent of  $B$  in  $\mathcal{W}$ .  $e$  introduces 1 extra vertex  $a_k$  in  $H$ . A vertex  $v$  is on  $\leq f_v(v)$  edges of sets  $I(B)$  (since these edges are in  $F_v$ ) so the total number of edges  $e \in \bigcup \{I(B) : B \in \mathcal{W}\}$  is  $\leq \phi$ .

## 9 Shortest-path tree algorithms

We construct the gsp-tree from the weighted blossom forest  $\mathcal{W}$  found in Section 8. This section shows how to find a gsp-tree for the graph  $\overline{G}(B)$ ,  $B$  a weighted blossom. It is a simple matter to join these gsp-trees together to get the entire gsp-tree for the given graph. (Note the root of the gsp-tree for any blossom  $B$  is a cycle node, not a tree node.) Let us restate the properties of  $\overline{G}(B)$ . Every edge is tight. There is a sink vertex  $t$ . Some vertices  $v$  have a distinguished edge  $e(v) \in \delta(v)$ , known to be on the shortest  $vt$ -path. (These vertices are children of  $B$  that are contracted blossoms, possibly singleton blossoms.) The remaining vertices of  $\overline{G}(B)$  have no such edge (these vertices are children of  $B$  that are vertices in the given graph). We wish to find a gsp-tree for this graph  $\overline{G}(B)$ .

We solve a slightly more general problem. Consider a graph with a sink vertex  $t$  and with every vertex  $v$  having a value  $e_0(v) \in \delta(v) \cup \{\emptyset\}$ ,  $e_0(t) = \emptyset$ . A  $vv'$ -path  $P$  is *permissible* if every vertex  $x \in P - v'$  has  $e_0(x) \in E(P) \cup \{\emptyset\}$ . In particular  $P$  starts with  $e_0(v)$  if it is nonnull. Observe that any  $vt$ -path specified by a gsp-tree is permissible for  $e$ . We will present an algorithm that, for a given  $G, t, e_0$ , finds a gsp-tree  $\mathcal{T}$  whose function  $e$  agrees with  $e_0$  on vertices where  $e_0 \neq \emptyset$ . In other words we will find a gsp-tree that specifies a permissible  $vt$ -path for every vertex  $v$  of  $G$ . For

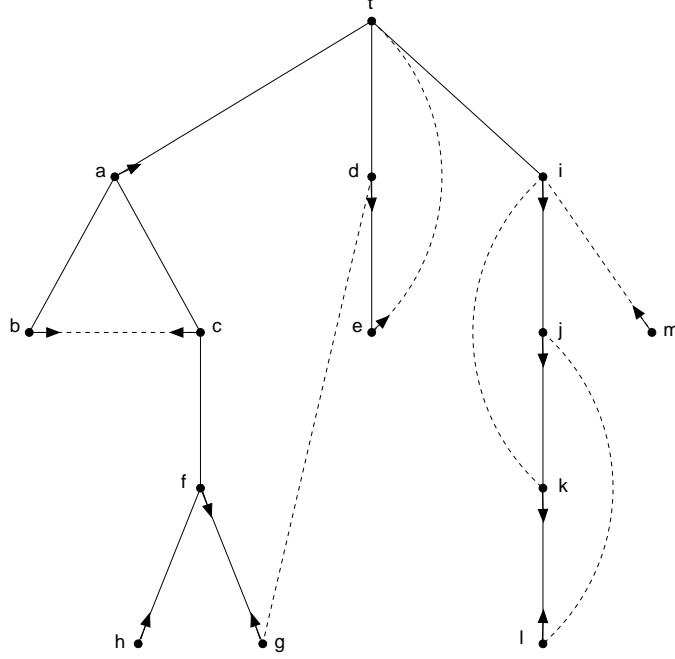


Figure 6: Illustration for permissible paths: Arrows indicate  $e(v)$  edges. Solid edges belong to a search tree. The 3 leftmost dashed edges cause contractions; the 3 rightmost cause no action.

convenience we assume the desired gsp-tree exists, as it does in our application.

To illustrate the discussion Fig.6 shows a given graph (all  $e_0$  values are edges). Vertices  $i, j, k, m$  have no permissible path to  $t$ . Fig.7 gives a gsp-tree for the subgraph of Fig.6 induced by  $t$  and  $a, \dots, h$ .

We begin with some definitions and facts that lay the foundations of the algorithm. The discussion refers to both the given function  $e_0$  and the final function  $e$ . In situations where either function can be used we try to use the more informative  $e_0$ . Also note that the notion of permissibility is slightly different for the two functions. We will specify the function  $e_0, e$  when it may not be clear.

A cycle  $C$  is *permissible* (for  $e$ ) if for some vertex  $b \in C$ , every vertex  $x \in C - b$  has  $e(x) \in E(C)$ ; furthermore,  $b = t$  if  $t \in C$  and  $b \neq t$  implies  $e(b) \in \delta(C)$ . Observe that for a permissible cycle  $C$ , any  $x \in C - b$  has a permissible path to  $b$  using edges of  $C$ .

We will find the desired gsp-tree  $\mathcal{T}$  by growing a “search tree”, repeatedly contracting a permissible cycle  $C$ , and making  $C$  a cycle node of  $\mathcal{T}$ . If  $\overline{C}$  denotes the contracted vertex, the contracted graph has  $e(\overline{C}) = e(b)$ . Also if  $b = t$  then the contracted graph has  $t = \overline{C}$ .

As an example in Fig.6 we will contract  $a, b, c$ ; this gives the leftmost cycle node of Fig.7; the contracted vertex gets  $e$ -value  $at$ .

Let  $\overline{G}$  be a graph formed from  $G$  by zero or more such contractions. A *search tree*<sup>15</sup>  $T$  is a tree in  $\overline{G}$  rooted at  $t$ , with each of its nodes  $v$  having one of two types, “u” for “up” or “d” for “down”, depending on where we find the edge  $e(v)$ . In precise terms let  $f$  be the edge from the parent of  $v$  to  $v$ . If  $f = e(v)$ , or if  $v = t$ , then  $v$  is a *u-vertex*. In the remaining case (i.e.,  $v \neq t$  and  $f \neq e(v)$ )  $v$

<sup>15</sup>The reader will recognize the resemblance to search trees used in cardinality matching algorithms.

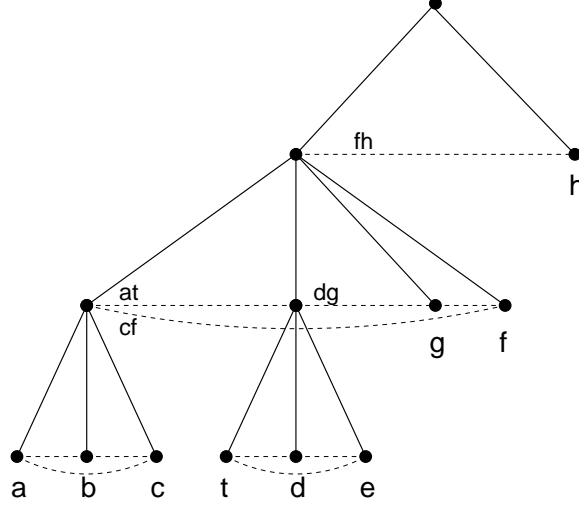


Figure 7: Gsp-tree for vertices  $a, \dots, h$  and  $t$  of Fig.6.  $e(N)$  edges are the dashed edges joining the children of  $N$ .

is a  $d$ -vertex. We require that a  $d$ -vertex  $v$  has at most one child, namely  $c$  where  $e(v) = vc$ .

As an example in Fig.6 a search tree might initially contain  $u$ -vertices  $t, a$  and  $d$ -vertices  $b, c$ . Contracting  $a, b, c$  makes the contracted vertex a  $u$ -vertex. Edge  $cf$  can be added to the search tree after this contraction, but not before. Adding  $cf$  makes  $f$  a  $d$ -vertex.

A search tree has the following property:

**Lemma 42.** *Consider a nontree edge  $x_1x_2$  where for  $i = 1, 2$ ,  $x_i$  is a  $u$ -vertex or  $x_i$  is a  $d$ -vertex having  $e(x_i) = x_1x_2$ . Let  $b$  be the nearest common ancestor of  $x_1, x_2$  and let  $C$  be the fundamental cycle of  $x_1x_2$ . Then  $C, b$  is a permissible cycle.*

Fig.6 illustrates the lemma. In the given graph (i.e., before any cycles have been contracted) edges  $bc$  and  $et$  give permissible cycles (for an appropriate search tree).  $bc$  ( $et$ ) illustrates the lemma when  $\{x_1, x_2\}$  contains 2 (1)  $d$ -vertices, respectively. After these 2 permissible cycles have been contracted,  $gd$  illustrates the lemma when  $\{x_1, x_2\}$  contains 2  $u$ -vertices. The nontree edge  $ik$  ( $jl$ ) illustrates a nonpermissible cycle, when  $\{x_1, x_2\}$  contains 2 (1)  $d$ -vertices, respectively.

*Proof.* First note  $b$  is a  $u$ -vertex. (If  $b$  is a  $d$ -vertex, it has just one child  $c$ , with  $e(b) = bc$ .  $x_1x_2$  must be a back edge with some  $x_i = b$ . The hypothesis makes  $e(x_i) = e(b) = x_1x_2$ , contradiction.) Now it is easy to check the properties for permissibility (e.g., a  $d$ -vertex  $v \in C$  has  $e(v) \in C$  since  $v$  is either an  $x_i$  or the edge from  $v$  to its unique child is in  $C$ ). ■

The algorithm will contract such cycles  $C$ , and only these. In particular this implies that in  $\overline{G}$ , any  $d$ -vertex is actually a vertex of  $G$ .

Also note the algorithm will choose values  $e(v)$  for vertices with  $e_0(v) = \emptyset$  by making  $v$  a  $u$ -vertex. In particular this implies that  $e(v) = e_0(v)$  for every  $d$ -vertex  $v$ .

**Algorithm** We will build the desired gsp-tree by growing a search tree  $T$ . Initialize  $T$  to consist of just a root  $t$  (a  $u$ -vertex) and let  $e$  be the given function  $e_0$ . Then repeat Algorithm 3 until every  $u$ -vertex  $x$  in the current graph has every incident edge either in  $T$  or scanned from  $x$ , and every  $d$ -vertex has been scanned.

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**Algorithm 3** A procedure that scans vertices and builds the gsp-tree.

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1: if some  $u$ -vertex  $x$  has an unscanned edge  $xy \notin T$  then                                 $\triangleright$  scan  $xy$  from  $x$ 
2:   if  $y \notin T$  then
3:     add  $xy$  to  $T$ 
4:   else if  $y$  is a  $u$ -vertex or  $xy = e_0(y)$  then                                 $\triangleright y$  is a  $d$ -vertex leaf
5:     contract the fundamental cycle of  $xy$ 
6:   end if
7: else if there is an unscanned  $d$ -vertex  $x$  then                                 $\triangleright$  scan  $x$ 
8:   let  $e_0(x) = xy$ 
9:   if  $y \notin T$  then
10:    add  $xy$  to  $T$ 
11:   else if  $xy = e_0(y)$  then                                 $\triangleright y$  is a  $d$ -vertex leaf
12:    contract the fundamental cycle of  $xy$ 
13:   end if
14: end if

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**Addendum** Several aspects of this algorithm are stated at a high level and deserve further elucidation:

An edge  $xy$  that is scanned from a  $u$ -vertex  $x$  is considered scanned from any  $u$ -vertex  $\bar{x}$  that contains  $x$  by contractions.

As mentioned, when the algorithm adds an edge  $xy$  to  $T$ , if  $e_0(y) = \emptyset$  it sets  $e(y) = xy$ . Furthermore in all cases it makes  $y$  type  $u$  or  $d$  as appropriate.

When the algorithm contracts a fundamental cycle  $C$ , it creates a node in  $\mathcal{T}$  whose children correspond to the vertices of  $C$ . If the algorithm halts with  $T$  containing nodes other than  $t$ , it creates a root node of  $\mathcal{T}$  whose children correspond to the vertices of  $T$ .

**Examples** Note that over the entire algorithm an edge may get scanned twice, once from each end, e.g., in Fig.6  $gd$  may get scanned from  $g$  when  $d$  is still a  $d$ -vertex. In a similar vein, if we change  $e_0(i)$  to a new edge  $id$ , the algorithm might scan  $i$  (and edge  $id$ ) before  $d$  becomes a  $u$ -vertex and  $id$  gets scanned from it.

**Analysis of the algorithm** The algorithm takes no action for the following types of edges  $xy$ :

- (i)  $x$  a  $u$ -vertex,  $y$  a  $d$ -vertex,  $xy \neq e_0(y)$ ,
- (ii)  $x$  and  $y$   $d$ -vertices,  $xy$  is not both  $e_0(x)$  and  $e_0(y)$ .
- (iii)  $x$  a  $d$ -vertex,  $y \notin T$ ,  $xy \neq e_0(x)$ .

(ii) and (iii) include edges that are never even scanned (e.g.,  $ik, im$  in Fig.6). When the algorithm halts every nontree edge of  $\overline{G}$  with an end in  $T$  is of type (i), (ii) or (iii).

Now we show the algorithm is correct. First note Lemma 42 shows every contracted cycle is permissible.

Assume every vertex  $v$  has a permissible path to  $t$  in the given graph  $G$  with function  $e_0$ . Let  $\overline{G}$  be the final graph of the algorithm. Let  $T$  be the final tree.

**Lemma 43.**  $\overline{G}$  consists entirely of  $u$ -vertices.

*Proof.* Let  $v$  be a non- $u$ -vertex, with  $P$  a permissible  $vt$ -path in  $G$ . The reader should bear in mind the possibility that the image of  $P$  in  $\overline{G}$  needn't be permissible (for  $e$ ). For instance suppose  $\overline{G}$  contains a contracted node like  $X = \{a, \dots, g, t\}$  in Fig.6. If the graph has two edges  $b'b, c'c \in \delta(X)$ ,  $P$  might contain a subpath  $b', b, c, c'$ , making it nonpermissible in  $\overline{G}$ . In fact similar edges incident to  $fg$  might make the image of  $P$  nonsimple in  $\overline{G}$ .

First observe that we can assume  $v$  is a  $d$ -vertex and  $V(P) \subseteq V(T)$ .<sup>16</sup> In proof, suppose  $P$  contains a nontree vertex (if not we're clearly done). It is eventually followed by an edge  $rs$ , where  $r \notin V(T)$  and  $V(Q) \subseteq V(T)$  for  $Q$  the  $st$ -subpath of  $P$ . (i)–(iii) show  $s$  is not a  $u$ -vertex. So  $s$  is a  $d$ -vertex. (iii) shows  $rs \neq e_0(s)$ , so  $Q$  is permissible.

Imagine traversing the edges of  $P$ , starting from  $v$ . Some edges of  $P$  will be in contracted vertices of  $\overline{G}$ , others will be edges of  $T$ , and all others will be of type (i) or (ii), not (iii) (since  $V(P) \subseteq V(T)$ ). We assert that whenever we reach a vertex  $r$  of  $\overline{G}$ , either

- (a)  $r$  is a  $d$ -vertex, and  $e_0(r)$  has not been traversed, or
- (b)  $r$  is a  $u$ -vertex, reached by traversing edge  $e(r)$ .

We prove the assertion by induction. Note that the assertion completes the proof of the lemma: (b) shows  $P$  always enters a  $u$ -vertex from its parent, so it never reaches the  $u$ -vertex  $t$ , contradiction.

For the base case of the induction,  $r = v$  obviously satisfies (a). For the inductive step assume the assertion holds for  $r$  and let  $s$  be the next vertex of  $\overline{G}$  that is reached. So  $rs$  is the next edge of  $\overline{G}$  that is traversed. ( $r$  or  $s$  may be contracted vertices of  $\overline{G}$ .)

**Case  $r$  satisfies (a):** Permissibility in  $G$  implies the next edge of  $P$  in  $\overline{G}$  is  $rs = e_0(r)$ .

Suppose  $e_0(r) \in T$ . This implies  $s$  is the child of  $r$ . The definition of search tree implies  $s$  satisfies (a) or (b), depending on whether or not  $rs = e_0(s)$ .

Suppose  $e_0(r) \notin T$ .  $e_0(r)$  is not type (i) above (even if we take  $r = y$ ). So it is type (ii). This implies (a) holds for  $s$ .

**Case  $r$  satisfies (b):** The argument is similar.

Suppose  $rs \in T$ . (b) shows  $r$  is reached from its parent in  $T$ . So  $s$  is a child of  $r$ . As in the previous case,  $s$  satisfies (a) or (b).

Suppose  $rs \notin T$ . So  $rs$  is type (i) above and (a) holds for  $s$ . ■

The lemma implies  $T$  with its contractions gives the desired gsp-tree  $\mathcal{T}$ . (E.g., if  $T$  consists of just one vertex  $t$ , the root of  $\mathcal{T}$  is a cycle node, specifically the last cycle to be contracted; otherwise the root is a tree node and its tree is  $T$ .)

It is easy to implement the above procedure in time  $O(m \log n)$  or better using an algorithm for set merging [11].

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<sup>16</sup>Here we commit a slight abuse of notation: The inclusion is meant to allow the possibility that a vertex of  $P$  is contained in a contracted vertex of  $T$ .

## 10 Combinatoric algorithms for shortest paths

Let  $(G, t, w)$  denote a connected undirected graph with a distinguished vertex  $t$  and a conservative edge-weight function  $w : E \rightarrow \mathbb{R}$ . Let  $E^-$  be the set of edges with negative weights. In this section we show how to use combinatoric algorithms for finding maximum perfect matchings to compute the gsp-tree. We will define a graph  $\ddot{G}_t$  that models paths in  $G$  by *almost perfect matchings*, i.e., matchings that miss exactly one vertex. Moreover, we define a *v-matching* to be an almost perfect matching in  $G$  that avoids  $v$ . We believe that the construction of the split graph  $\ddot{G}$  is essentially due to Edmonds [9]. We define the *split graph*  $\ddot{G} = (\ddot{V}, \ddot{E})$  with weight function  $\ddot{w}$  in the following way

$$\begin{aligned}\ddot{V} &= \{v_1, v_2 : v \in V\} \cup \{e_1, e_2 : e \in E^-\}, \\ \ddot{E} &= \{v_1 v_2 : v \in V\} \cup \{u_1 v_2, u_2 v_1, u_1 v_1, u_2 v_2 : uv \in E \setminus E^-\} \\ &\quad \cup \{u_1 e_1, u_2 e_1, e_1 e_2, v_1 e_2, v_2 e_2 : e = uv \in E^-, u < v\}, \\ \ddot{w}(u_i v_j) &= \begin{cases} -w(uv) & \text{if } uv \in E \setminus E^-, \\ -w(e) & \text{if } u_i = e_1 \text{ and } v_j \neq e_2 \text{ and } e \in E^-, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

An important property is that we can assume  $\ddot{n} = |\ddot{V}| \leq 4n$ . This follows since we can assume  $|E^-| < n$ , as otherwise the set of negative edges contains a cycle. The following observation is essentially given in [1] in Chapter 12.7, where the reduction is explained on a clear example.

**Lemma 44.** *Let  $u, v \in V$ , let  $M$  be the maximum perfect matching, and let  $M(u_2 v_1)$  be the maximum weight perfect matching in  $\ddot{G} - u_2 - v_1$ . If  $G$  does not contain negative weight cycles then  $\ddot{w}(M) = 0$  and the shortest path weight from  $u$  to  $v$  in  $G$  is equal to  $-\ddot{w}(M(u_2 v_1))$ .*

Note also that it is easy to detect a negative cycle in  $G$  – it corresponds to a perfect matching in  $\ddot{G}$  with positive weight. On the other hand, as described in [1], in order to find a shortest path from  $u$  to  $v$  we need to find the maximum weight perfect matching  $M(u_2 v_1)$ . However, here we want to compute the whole gsp-tree and hence require the distances from all vertices in  $G$  to  $t$ . We will show that in order to find all these distances it is essentially enough to find one maximum perfect matching. Let us define  $\ddot{G}_t$  to be graph  $\ddot{G}$  with both vertices  $t_1$  and  $t_2$  unified to one vertex  $t$ . We observe that the resulting graph is *critical*, i.e., for each vertex  $v$  there exists a *v-matching*.

**Lemma 45.** *Graph  $\ddot{G}_t$  is critical. Moreover, let  $M(v_1)$  be the maximum  $v_1$ -matching in  $\ddot{G}_t$  then  $\ddot{w}(M(t_2 v_1)) = \ddot{w}(M(v_1))$ .*

*Proof.* We need to show the existence of *v-matchings* for all vertices  $v$  in  $\ddot{G}_t$ . Consider the following cases

- $v = t$  then  $M(t_2 t_1)$  in  $\ddot{G}$  corresponds matching that avoids  $t$ ,
- $v = v_2$  then  $M(t_1 v_2)$  in  $\ddot{G}$  corresponds to matching that avoids  $v_2$ ,
- $v = v_1$  then in this case we take  $M(t_2 v_1)$  in  $\ddot{G}$ .

Hence,  $\ddot{G}_t$  is critical. The second part of the lemma follows by the above correspondence of matchings. ■

We need to relate the above observation to the definitions from previous sections. You might observe that  $M(v_1)$  corresponds to  $P_v$  from Section 6 and in language of  $f$ -factors to  $F_v$  as discussed in Section 8. In order to construct the gsp-tree we use the shrinking procedure for general  $f$ -factors from Section 9. This requires us to know  $w(F^v)$  as well. However, we can observe that  $F^v$  can be obtained from  $F_v$  and vice versa by adding or removing the zero weight loop  $vv$ .

## 10.1 Matching duals for factor critical graphs

Consider an arbitrary graph  $G = (V, E)$  that contains a perfect matching. Let  $w : E \rightarrow \mathbb{R}$  be the edge weight function. The dual variables in Edmonds' formulation [8] are assigned to vertices  $y : V \rightarrow \mathbb{R}$  and to odd-size subsets of vertices  $z : 2^V \rightarrow \mathbb{R}$ . The function  $z$  can be negative possibly only on  $V$ . We define the value of  $uw$  (with respect to the dual  $y, z$ ) as

$$\widehat{yz}(uv) = y(u) + y(v) + z\{B : e \subseteq B\}.$$

We require the duals to *dominate* all edges  $uv \in E$ , i.e., we require

$$\widehat{yz}(uv) \geq w(uv).$$

We say that an edge is *tight* when the above inequality is satisfied with equality. The *dual objective* is defined as

$$(y, z)V = y(V) + \sum \{ \lfloor |B|/2 \rfloor z(B) : B \subseteq V \}.$$

By the duality we know that for any perfect matching  $M$  and duals  $y, z$  we have  $w(M) \leq (y, z)V$ . We say that a matching *respects* a set  $B$  if it contains  $\lfloor |B|/2 \rfloor$  edges in  $\gamma(B)$ . As shown by Edmonds [8] a perfect matching is maximum if and only if all its edges are tight and it respects all sets with positive  $z$  for a pair of dominating duals.

A *blossom* is a subgraph  $B$  of  $G$  defined as follows. Every vertex is a blossom and has no edges. Otherwise, the vertices  $V(B)$  are partitioned into an odd number  $k$  of sets  $V(B_i)$ , for  $1 \leq i \leq k$ , where each  $B_i$  is a blossom. Each blossom  $B$  contains edges in  $E(B_i)$  and  $k$  edges that form a cycle on  $B_i$ , i.e., consecutive edges end in  $V(B_i)$  and  $V(B_{i+1})$ , where  $B_1 = B_{i+1}$ . Blossoms  $B_i$  are called subblossoms of  $B$ . The set of blossoms can be represented as forrest called *blossom forrest*. The parent-child relation in this forrest is defined by the blossom-subblossom relation.

Edmonds' algorithm for finding maximum perfect matchings constructs a *structured matching*, i.e., a matching  $M$  and a dual solution. The dual solution is composed out of a blossom forrest  $F$  and functions  $y$  and  $z$ . The structured matching satisfies the following conditions

- (i)  $M$  respects blossoms in  $F$ ,
- (ii)  $z$  is nonzero on blossoms in  $F$ ,
- (iii) all edges in  $M$  are tight,
- (iv) all edges in blossoms in  $F$  are tight.

If one finds a structured matching that is perfect then it is a maximum perfect matchings. In the classical view Edmonds' algorithm operates on a graph with even number of vertices. However, as shown in [15] it can be seen to work on a critical graph. The *optimal matching structure* of a critical graph  $G$  consists a blossom tree  $B$  and dual functions  $y, z$  such that every vertex is a leaf in  $B$  and properties (ii) and (iv) above are satisfied. Observe that there is no matching in this definition. As argued in [15] Edmonds' algorithm computes an optimum matching structure when it is executed on a critical graph. Moreover, the optimal matching structure allow us to relate weights of maximum  $v$ -matchings to the dual  $y, z$  in the following way.

**Lemma 46** ([15]). *Let  $y, z$  and  $B$  the optimal matching structure for the critical graph  $G$ . Then the weight of maximum  $v$ -matching is to equal to  $(y, z)V - y(v)$ .*

## 10.2 The algorithm

Let us now join all the ingredients to compute the gsp-tree in the undirected graph  $(G, w, t)$  with the conservative weight function  $w$ .

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**Algorithm 4** A combinatoric algorithm for computing gsp-tree for the undirected graph  $(G, w, t)$ .

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- 1: Construct  $\ddot{G}$  from  $G$
  - 2: Construct  $\ddot{G}_t$  from  $\ddot{G}$  by identifying  $t_1$  and  $t_2$
  - 3: Compute optimal matching structure  $y, z$  and  $B$  for  $\ddot{G}_t$  ▷  $\ddot{G}_t$  is critical by Lemma 45
  - 4: **for**  $v \in V$  **do**
  - 5:   Let  $w(M_v) = (y, z)V - y(v)$  ▷ by Lemma 46
  - 6:   Let  $w(F_v) = -w(M_v)$  ▷ by Lemma 45 and Lemma 44
  - 7:   Let  $w(F^v) = w(F_v)$  ▷ by adding zero length loop  $vv$
  - 8: **end for**
  - 9: Using  $w(F_v)$  and  $w(F^v)$  find the blossom forrest ▷ using shrinking procedure from Section 8
  - 10: Construct gsp-tree from the blossom forrest ▷ using procedure from Section 9
- 

We note that optimal matching structure can be found using fast implementations of Edmonds' algorithm in  $O(n(m + n \log n))$  time [13] or in  $O(\sqrt{n\alpha(m, n)} \log n \cdot m \log(nW))$  time [15]. The other steps of the above algorithm take only less time, e.g., the shrinking procedure can be implemented to work in  $O(m \log n)$  time. This way we obtain the  $O(n(m + n \log n))$  time and the  $O(\sqrt{n\alpha(m, n)} \log n \cdot m \log(nW))$  time combinatoric algorithms for computing gsp-tree.

## 11 Determinant formulations for general graphs

Let  $G$  be a simple graph with vertices numbered from 1 to  $n$ . Let  $\phi = \sum_i f(i)$ . We define a skew-symmetric  $\phi \times \phi$  matrix  $B(G)$  representing  $G$  in the following way. A vertex  $i \in V$  is associated with  $f(i)$  consecutive rows and columns of  $B$ , both indexed by the pairs  $i, r$  for  $0 \leq r < f(i)$ . Call a tuple  $(i, r, j, c)$  representing the entry  $B(G)_{i,r,j,c}$  *permissible* if either

$$i < j, \text{ or } i = j \text{ and } r < \lfloor f(i)/2 \rfloor \leq \lceil f(i)/2 \rceil \leq c < f(i).$$

The set of permissible entries is denoted by  $\mathbb{P}$ . Note that the permissible entries of  $B(G)$  are all above the diagonal. The permissible entries corresponding to a fixed edge  $e$  form a rectangular submatrix for a nonloop  $e$  and an  $\lfloor f(i)/2 \rfloor \times \lfloor f(i)/2 \rfloor$  submatrix for a loop  $ii$ .

Using indeterminates  $x_r^{ij}, y_c^{ij}$  we define an entry of  $B(G)$  as

$$B(G)_{i,r,j,c} = \begin{cases} x_r^{ij} y_c^{ij} & ij \in E \text{ and } (i, r, j, c) \in \mathbb{P}, \\ -x_c^{ji} y_r^{ji} & ij \in E \text{ and } (j, c, i, r) \in \mathbb{P}, \\ 0 & \text{otherwise.} \end{cases} \quad (25)$$

Clearly  $B(G)$  is skew-symmetric. Note that a loop  $ii \in E$  is represented by an  $\lfloor f(i)/2 \rfloor \times \lfloor f(i)/2 \rfloor$  submatrix that is empty if  $f(i) = 1$ . This is fine since  $ii$  is not in any  $f$ -factor.

The analysis for multigraphs is almost identical to simple graphs. So we will concentrate on simple graphs, but also point out how it extends to multigraphs. Towards this end we extend the definition of  $B(G)$  as in Section 4.1: Let  $\mu(e)$  denote the multiplicity of any edge  $e$ . The copies of  $e = ij$  get indeterminates  $x_r^{ij,k}, y_c^{ij,k}$  ( $1 \leq k \leq \mu(e)$ ) and we define  $B(G)$  by

$$B(G)_{i,r,j,c} = \begin{cases} \sum_{k=1}^{\mu(e)} x_r^{ij,k} y_c^{ij,k} & ij \in E \text{ and } (i, r, j, c) \in \mathbb{P}, \\ -\sum_{k=1}^{\mu(e)} x_c^{ji,k} y_r^{ji,k} & ij \in E \text{ and } (j, c, i, r) \in \mathbb{P}, \\ 0 & \text{otherwise.} \end{cases} \quad (26)$$

### 11.1 Review of the Pfaffian

Our goal is to prove an analog of Theorem 5. It is easiest to accomplish this using the Pfaffian, so we begin by reviewing this concept. Let  $A$  be a skew-symmetric matrix of order  $2h \times 2h$ . Its Pfaffian is defined by

$$\text{pf}(A) = \sum \text{sgn} \begin{pmatrix} 1 & 2 & \dots & 2h-1 & 2h \\ i_1 & j_1 & \dots & i_h & j_h \end{pmatrix} a_{i_1 j_1} \dots a_{i_h j_h}.$$

The above sum is over all partitions of the integers  $[1..2h]$  into pairs denoted as  $\{i_1, j_1\}, \dots, \{i_h, j_h\}$ . Also  $\text{sgn}$  denotes the sign of the permutation. (It is easy to check that each term  $\sigma$  in the summation is well-defined. For example interchanging  $i_1$  and  $j_1$  does not change the partition. It flips both the sign of the permutation and the sign of  $a_{i_1 j_1}$ , so  $\sigma$  is unchanged.) The central property of the Pfaffian is [19]

$$\det(A) = (\text{pf}(A))^2. \quad (27)$$

Recall that the Tutte matrix  $T$  for a graph  $G$  is the skew-symmetric matrix obtained from the adjacency matrix of  $G$  by replacing the entry 1 for edge  $ij$  by an indeterminate  $t_{ij}$  if  $i < j$  and by  $-t_{ji}$  if  $i > j$ . Clearly there is a 1-1 correspondence between perfect matchings of  $G$  and terms of  $\text{pf}(T)$ .

Recalling (27), note that every pair of perfect matchings of  $G$  gets combined in the product  $(\text{pf}(T))^2$ . Let us review a proof that no cancellations occur when all these terms are added together.

Consider a term of  $(\text{pf}(T))^2$  corresponding to perfect matchings  $M_1, M_2$ . (These matchings may be distinct or identical.) The terms of  $(\text{pf}(T))^2$  that might cancel this term must use exactly the same variables  $t_{ij}$ . In other words they correspond to perfect matchings  $N_1, N_2$  where

$$M_1 \uplus M_2 = N_1 \uplus N_2. \quad (28)$$

Here  $\uplus$  denotes multiset sum.  $M_1 \uplus M_2$  consists of the edges of  $M_1 \cap M_2$  taken twice, plus the set  $M_1 \oplus M_2$ . The latter is a collection  $\mathcal{C}$  of even alternating cycles. The definition of matching shows every cycle of  $\mathcal{C}$  is alternating wrt  $N_1, N_2$ . Assume that in  $(\text{pf}(T))^2$ , the first multiplicand  $\text{pf}(T)$  gives the matching  $N_1$  and the second multiplicand gives  $N_2$ . The edges of each cycle of  $\mathcal{C}$  can be partitioned in two ways between  $N_1$  and  $N_2$ . So there are precisely  $2^{|\mathcal{C}|}$  matching pairs  $N_1, N_2$  of  $(\text{pf}(T))^2$  that satisfy (28).

We will show there are no cancellations because each matching pair  $N_1, N_2$  gives the same term of  $(\text{pf}(T))^2$ . More precisely we show the following:

**Claim 1.** *The pairs  $N_1, N_2$  satisfying (28) collectively contribute the quantity  $(-2)^{|\mathcal{C}|} \prod t_{ij}$  to  $(\text{pf}(T))^2$ , where the product is over a fixed set of entries in  $T$  corresponding to  $M_1 \uplus M_2$ .*

Note the claim also implies that if we do arithmetic over a finite field of characteristic  $> 2$ , again there are no cancellations.

*Proof of Claim 1.* Take a pair of matchings  $N_1, N_2$  satisfying (28). Let  $\sigma_i$  be the term in  $\text{pf}(T)$  for  $N_i$ , so  $(\text{pf}(T))^2$  contains  $\sigma_1\sigma_2$ . We will show the portion of  $\sigma_1\sigma_2$  corresponding to  $C$  is the same for every  $N_1, N_2$ . Then we will conclude this property makes the entire term  $\sigma_1\sigma_2$  independent of choice of  $N_1, N_2$ .

First suppose  $C$  is a single edge  $ij$  (belonging to  $M_1 \cap M_2$ ). Wlog both  $\sigma_1$  and  $\sigma_2$  have the  $T$  entry  $t_{ij}$  and their permutations both map some ordered pair of two consecutive integers  $(2a-1, 2a)$  to the ordered pair  $(i, j)$ . Clearly this does not depend on choice of  $N_1, N_2$ .

Now suppose  $C$  contains vertices  $i_1, \dots, i_\ell$  for some even  $\ell \geq 4$ , and  $N_1$  contains edges  $i_1i_2, \dots, i_{\ell-1}i_\ell$  while  $N_2$  contains  $i_2i_3, \dots, i_\ell i_1$ .  $\sigma_1$  contains the product  $t_{i_1i_2} \dots t_{i_{\ell-1}i_\ell}$  and  $\sigma_2$  contains  $t_{i_2i_3} \dots t_{i_\ell i_1}$ . So  $\sigma_1\sigma_2$  contains the product  $(t_{i_1i_2} \dots t_{i_{\ell-1}i_\ell})(t_{i_2i_3} \dots t_{i_\ell i_1})$ . Certainly this is independent of choice of  $N_1, N_2$ .

The permutation for  $\sigma_1$  maps some pair of two consecutive integers  $(2a-1, 2a)$  to  $(i_1, i_2)$ , and similarly for the rest, e.g.,  $(2b-1, 2b)$  goes to  $(i_{\ell-1}, i_\ell)$ . Wlog  $N_2$  maps  $(2a-1, 2a)$  to  $(i_2, i_3)$ , and similarly for the rest, e.g.,  $(2b-1, 2b)$  goes to  $(i_\ell, i_1)$ . The pairs of  $\sigma_1$  are transformed to the pairs of  $\sigma_2$  by applying the product of transpositions  $(i_1i_2)(i_1i_3) \dots (i_1i_\ell)$ . This is an odd number of transpositions, i.e., its sign is  $-1$ . So the part of the permutations for  $\sigma_1$  and  $\sigma_2$  in  $C$  combine to give the sign  $-1$  in  $\sigma_1\sigma_2$ .

Applying this analysis of sign to each of the  $|\mathcal{C}|$  components of  $\geq 4$  vertices shows the entire permutation of  $\sigma_1\sigma_2$  contributes sign  $(-1)^{|\mathcal{C}|}$ . Since there are  $2^{|\mathcal{C}|}$  pairs  $N_1, N_2$ , we get the contribution of the claim.  $\diamond$

We shall also use a special case of the above analysis, specifically when  $N_1$  and  $N_2$  are identical except for choosing alternate edges of one cycle of length 4.  $\sigma_1$  and  $\sigma_2$  have opposite sign, and so they differ only in the subexpressions

$$\pm t_{i_1i_2}t_{i_3i_4} \text{ and } \mp t_{i_2i_3}t_{i_4i_1} \quad (29)$$

for some consistent choice of sign.

## 11.2 Analysis of $B(G)$

We return to the matrix  $B(G)$  for  $f$ -factors. An entry of  $B(G)$  corresponds to an edge of  $G$ , so a term of  $\text{pf}(B(G))$  corresponds to a multiset of edges of  $G$ .

**Claim 2.** *Each uncanceled term  $\sigma$  in  $\text{pf}(B(G))$  corresponds to an  $f$ -factor of  $G$ .*

*Proof of Claim 2.* The above discussion shows that ignoring sign,  $\sigma$  is a product of  $\phi/2$  quantities  $x_r^{ij}y_c^{ij}$  corresponding to edges that form a matching on vertices designated by two indices  $i, r$  ( $1 \leq i \leq n$ ,  $0 \leq r < f(i)$ ). So each vertex  $i$  of  $G$  is on exactly  $f(i)$  of the  $\phi/2$  edges selected by  $\sigma$  (counting loops  $ii$  twice). To show these edges form an  $f$ -factor of  $G$  we must show that  $\sigma$  uses each edge of  $G$  at most once. We accomplish this by showing that in the summation of the Pfaffian, terms using an edge more than once cancel in pairs.

Suppose  $\sigma$  uses edges  $x_r^{ij}y_c^{ij}$  and  $x_{r'}^{ij}y_{c'}^{ij}$  ( $r \neq r'$  and  $c \neq c'$ ). Wlog assume  $(i, r; j, c), (i, r'; j, c') \in \mathbb{P}$ .  $\sigma$  may have many such duplicated pairs. Choose the duplicated pair that lexically minimizes  $(i, r, r')$ . Pair  $\sigma$  with the term  $\sigma'$  having the same partition except that it replaces  $\{i, r; j, c\}$  and  $\{i, r'; j, c'\}$  by

$\{i, r; j, c'\}$  and  $\{i, r'; j, c\}$ . Note that permissibility of  $(i, r; j, c)$  and  $(i, r'; j, c')$  implies permissibility of  $(i, r; j, c')$  and  $(i, r'; j, c)$ , even if  $i = j$ . Thus  $\sigma'$  is also associated with  $(i, r, r')$ , so the pairing is well-defined.

Viewed as matchings,  $\sigma$  and  $\sigma'$  differ only by choosing alternate edges of the length 4 cycle  $(i, r; j, c; i, r'; j, c')$ . So  $\sigma$  and  $\sigma'$  are identical except for the expressions of (29), which in the new setting become

$$\pm b_{i,r;j,c} b_{i,r';j,c'} \text{ and } \mp b_{j,c;i,r'} b_{j,c';i,r},$$

where  $b$  designates matrix  $B(G)$ . Substituting the definition of  $B(G)$  shows these expressions are

$$\pm(+x_r^{ij} y_c^{ij})(+x_{r'}^{ij} y_{c'}^{ij}) \text{ and } \mp(-x_r^{ij} y_c^{ij})(-x_{r'}^{ij} y_{c'}^{ij}).$$

The two minus signs in the second expression follow from permissibility of  $(i, r; j, c')$  and  $(i, r'; j, c)$ . The 4 indeterminates in the 2 above expressions are collectively identical. So we get  $\sigma = -\sigma'$ , i.e., these two terms cancel each other as desired.

This argument also applies to multigraphs  $G$ . Here the issue is that only one copy of each distinct edge  $ij, k$  can be used. A term  $\sigma$  with a duplicated edge, like  $x_r^{ij,k} y_c^{ij,k}$  and  $x_{r'}^{ij,k} y_{c'}^{ij,k}$ , chooses the duplicated pair to lexically minimize  $(i, k, r, r')$ . The rest of the argument is unchanged.  $\diamond$

Every  $f$ -factor  $F$  of  $G$  has an uncanceled term  $\sigma$  in  $\text{pf}(B(G))$ : For each  $i \in V$ , order the set  $\delta(i, F)$  arbitrarily. This makes  $F$  correspond to a term  $\sigma$  in  $\text{pf}(B(G))$ . Each variable  $x_r^{ij}, y_c^{ij}$  in  $\sigma$  specifies its edge, and so determines a unique partition pair  $\{i, r; j, c\}$ . So no other term of  $\text{pf}(B(G))$  has the same variables of  $\sigma$ , and  $\sigma$  is uncanceled, (Of course a given  $F$  gives rise to many different uncanceled terms.)

Now it is easy to see that (27) gives an analog of the second assertion of Theorem 5:  $G$  has an  $f$ -factor if and only if  $\det(B(G)) \neq 0$ . But we need to prove the stronger first assertion.

We extend the above construction for  $F$  to an arbitrary pair of  $f$ -factors  $F_1, F_2$ . As before for each  $i \in V$  and for each  $F_j$  ( $j = 1, 2$ ), number the edges of  $\delta(i, F_j)$  (consecutively, starting at 1). But now for both sets start with the edges of  $\delta(i, F_1 \cap F_2)$ , using the same numbering for both. Using both these numberings gives two terms in  $\text{pf}(B(G))$ , which combine to give a term  $\sigma$  of  $(\text{pf}(B(G)))^2$ . Each variable  $x_r^{ij}$  in  $\sigma$  specifies its edge and corresponds to a unique  $y_c^{ij}$ . Here we are using the fact that an edge in both  $f$ -factors gives rise to a subexpression of the form  $(x_r^{ij} y_c^{ij})^2$  in  $\sigma$ .

Call a term of  $(\text{pf}(B(G)))^2$  *consistent* if whenever an edge  $ij$  occurs twice, it uses the same subscripts, thus giving a product of the form  $(x_r^{ij} y_c^{ij})^2$ . (We are disqualifying terms with an edge appearing as both  $x_r^{ij} y_c^{ij}$  and also  $x_{r'}^{ij} y_{c'}^{ij}$  where  $r \neq r'$  or  $c \neq c'$ .) Clearly the above construction gives consistent terms. We will show there is no cancellation in  $(\text{pf}(B(G)))^2$  involving consistent terms. (After that we comment on inconsistent terms.)

The proof follows the Tutte matrix argument: Consider a pair of  $f$ -factors  $F_1, F_2$ , and some consistent term  $\sigma$  in  $(\text{pf}(B(G)))^2$  that corresponds to them. View these  $f$ -factors as matchings  $FM_1, FM_2$  on vertices  $i, r$ . Another pair of  $f$ -factors  $H_1, H_2$  involves the same set of indeterminates exactly when  $FM_1 \uplus FM_2 = HM_1 \uplus HM_2$ . (Here we use the fact that by consistency, each indeterminate  $x_r^{ij}$  specifies its edge  $ij$  as well as its corresponding indeterminate  $y_c^{ij}$ . Hence the indeterminates in a term determine the matched edges  $(i, r; j, c)$ .) We have now established the analog of (28). The rest of the argument for Tutte matrices applies unchanged. Furthermore exactly the same analysis applies when  $G$  is a multigraph.

We comment that the situation is more involved for an inconsistent term  $\sigma$ . Suppose  $F_1$  has Pfaffian term  $\sigma_1$  containing  $x_r^{ij} y_c^{ij}$  and  $F_2$  has  $\sigma_2$  containing  $x_{r'}^{ij} y_{c'}^{ij}$ , where  $r \neq r'$  and  $c \neq c'$ .

Furthermore suppose the two copies of  $ij$  belong to different cycles  $C_1, C_2$  of  $FM_1 \uplus FM_2$ . There can be many different possibilities for two other  $f$ -factors  $H_1, H_2$  having  $HM_1 \uplus HM_2$ . For instance suppose edge  $ij$  gives the only inconsistency in  $\sigma_1\sigma_2$ . Then  $C_1$  and  $C_2$  only give rise to 2 possible matchings  $HM_1, HM_2$  instead of 4, since an  $f$ -factor cannot contain both copies of  $ij$ . But if there are inconsistent edges besides  $ij$ , other partitions for the edges of  $C_1$  and  $C_2$  may be possible (e.g., consider the case of 1 other inconsistent edge, with copies in  $C_1$  and  $C_2$ ).

As in Section 4 define  $\mathcal{F}$  to be the function that maps each term  $\sigma$  of  $\det(B(G))$  to its corresponding subgraph denoted  $F_\sigma$ .  $F_\sigma$  is a  $2f$ -factor (unlike Section 4). Let  $\Phi_2(G)$  be the set formed by taking sums of two (possibly equal)  $f$ -factors of  $G$ , i.e.,

$$\Phi_2(G) = \{F_1 \uplus F_2 : F_1, F_2 \text{ } f\text{-factors of } G\}.$$

We have proved the following.

**Theorem 47.** *Let  $G$  be a simple graph or a multigraph. The function  $\mathcal{F}$  from terms in  $\det(B(G))$  is a surjection onto  $\Phi_2(G)$ . Consequently,  $G$  has an  $f$ -factor if and only if  $\det(B(G)) \neq 0$ .*

The theorem continues to hold when we do arithmetic in any finite field of characteristic  $> 2$ .

## 12 Finding $f$ -factors in general graphs

This section gives algorithms to find  $f$ -factors in general multigraphs. It starts with simple graphs and then moves to multigraphs. One should keep in mind that it is unknown how to use Gaussian elimination in the non-bipartite case, e.g., [21] uses a different non-algebraic algorithm for this case. However Harvey [17] later developed a fully algebraic scheme, that we adopt here. A good explanation of this approach is given in [17]. We first use the Sherman-Morrison formula to get an  $O(m\phi^2)$  time algorithm. Then we show it can be sped up to  $O(\phi^\omega)$  time using ideas from Harvey's recursive elimination scheme.

### 12.1 Simple Graphs

We define *removable* edge  $ij \in E(G)$  to be an edge such that  $G - ij$  has an  $f$ -factor. We can observe that following property.

**Corollary 48.** *Let  $G$  be a simple graph having an  $f$ -factor. The edge  $ij \in E(G)$  is removable if and only if  $\det(B(G - ij)) \neq 0$ .*

Let  $ij \in E(G)$  then we can observe that  $B(G)$  and  $B(G - ij)$  differ from one another by two rank-one updates.

$$B(G - ij) = B(G) - x^{ij}(y^{ij})^T + y^{ij}(x^{ij})^T = B(G) + [-x^{ij}, y^{ij}][y^{ij}, x^{ij}]^T. \quad (30)$$

where  $x^{ij}$  and  $y^{ij}$  are length  $\phi$  vectors. When  $i \neq j$  these vectors are defined as

$$x_{k,r}^{ij} = \begin{cases} x_r^{ij} & \text{if } k = j, \\ 0 & \text{otherwise,} \end{cases} \quad y_{k,r}^{ij} = \begin{cases} y_r^{ij} & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand for  $ii \in E(G)$  we have

$$x_{k,r}^{ii} = \begin{cases} x_r^{ii} & \text{if } k = i \text{ and } r < \lfloor f(i)/2 \rfloor, \\ 0 & \text{otherwise,} \end{cases} \quad y_{k,r}^{ii} = \begin{cases} y_r^{ii} & \text{if } k = i \text{ and } f(j)/2 \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, one can use Sherman-Morrison-Woodbury formula to compute  $B(G-ij)^{-1}$  from  $B(G)^{-1}$  in  $O(\phi^2)$  time. Similarly, we can use this formula to test whether  $B(G-ij)$  is nonsingular by checking

$$\det(I_2 + [y^{ij}, x^{ij}]^T B(G)^{-1} [-x^{ij}, y^{ij}]) \neq 0. \quad (31)$$

Here, the affected matrix size is bounded by  $2f(i) + 2f(j)$  so we need  $O((f(i) + f(j))^2)$  time for this test. Using these observations we get Algorithm 5 for finding  $f$ -factors in simple graphs.

---

**Algorithm 5** An  $O(m\phi^2)$  time algorithm for finding  $f$ -factor in the simple graph  $G$ .

---

- 1: Let  $B(G)$  be  $\phi \times \phi$  skew-symmetric adjacency matrix of  $G$
  - 2: Replace the variables in  $B(G)$  for random elements from  $\mathcal{Z}_p$  for prime  $p = \Theta(\phi^2)$  to obtain  $B$
  - 3: If  $B$  is singular return "no  $f$ -factor".
  - 4: (with probability  $\geq 1 - \frac{1}{\phi}$  matrix  $B$  is non-singular when  $B(G)$  is non-singular)  $\triangleright$  by Lemma 1
  - 5: Compute  $B^{-1}$
  - 6: (we remove only removable edges so  $B$  remains non-singular during execution of the algorithm)
  - 7: **for all**  $ij \in E$  **do**
  - 8:     **if**  $\det(I_2 + [y^{ij}, x^{ij}]^T B^{-1} [-x^{ij}, y^{ij}]) \neq 0$  **then**  $\triangleright$  Edge  $ij$  is removable by (31)
  - 9:         Set  $E := E - e$
  - 10:         Set  $B := B + [-x^{ij}, y^{ij}] \cdot [y^{ij}, x^{ij}]^T$   $\triangleright$  This corresponds to  $B(G) - ij$  by (30)
  - 11:         Recompute  $B^{-1}$   $\triangleright$  Using Sherman-Morrison-Woodbury formula
  - 12:     **end if**
  - 13: **end for**
  - 14: Return  $E$   $\triangleright$  All removable edges have been removed, so what remains is an  $f$ -factor
- 

Observe that the above algorithm fits into the framework introduced by Harvey [17] for finding 1-factors in general graphs. The above algorithm corresponds to the algorithm described in Section 3.3 from [17], with the difference that the rank-two updates we use affect submatrices and not single elements. This means that we can use the recursive algorithm that was introduced in Section 3.4 of his paper. The algorithm uses following elimination procedures with the starting call to `DeleteEdgesWithin(V)`.

`DeleteEdgesWithin(S)` – if  $|S| \geq 1$  split  $S$  into  $S_1$  and  $S_2$ ; call `DeleteEdgesWithin(Si)`, for  $i = 1, 2$ ; call `DeleteEdgesCrossing(S1, S2)`; update submatrix  $B^{-1}[S, S]$ ;

`DeleteEdgesCrossing(R, S)` – if  $|R| = \{r\}$ ,  $|S| = \{s\}$  and  $rs$  is removable eliminate edge  $rs$ ; otherwise split  $R$  into  $R_1, R_2$  and  $S$  into  $S_1, S_2$ ; call `DeleteEdgesCrossing(Ri, Sj)`, for  $i, j = 1, 2$ ; update submatrix  $B^{-1}[R \cup S, R \cup S]$ .

Observe that we need only to replace tests for removable edges with (31), whereas the updates to submatrices remain essentially the same and use [17, Corollary 2.1]. Hence, the submatrix of size  $|S| \times |S|$  is updated in  $O(|f(S)|^\omega)$  time. On the other hand, the cost we pay to test whether the edge  $ij$  is removable is  $O(|f(i) + f(j)|^2)$ .

Harvey splits the set  $S$  (similarly  $R$ ) always in equal halves. We, however, split  $S$  in such a way that  $f(S_1)$  and  $f(S_2)$  are as close as possible. Let us assume  $f(S_1) \geq f(S_2)$  then

- either  $f(S_1) \geq \frac{2}{3}f(S)$  and  $|S_1| = 1$ ,
- or  $f(S_1) \leq \frac{2}{3}f(S)$  and  $f(S_2) \geq \frac{1}{3}f(S)$ .

Let  $h(S)$  denote the running time of the procedure `DeleteEdgesWithin( $S$ )`. Similarly define  $g(R, S)$  for `DeleteEdgesCrossing( $R, S$ )`. We have

$$\begin{aligned} h(S) &= \sum_i h(S_i) + g(S_1, S_2) + O(|f(S)|^\omega) \\ g(R, S) &= \begin{cases} O(|f(R) + f(S)|^2) & \text{if } |R| = |S| = 1, \\ \sum_{i,j} g(R_i, S_j) + O(|f(R) + f(S)|^\omega) & \text{otherwise.} \end{cases} \end{aligned}$$

The solution for these equations gives an  $O(\phi^\omega)$  time bound.

## 12.2 Multigraphs

In the case of multigraphs we need to handle multiple copies of the same edge in a different way. Removing separate copies of an edge one by one would lead to a cubic time complexity. The cost charged by edge  $ij$  to its submatrix would be  $\Omega((f(i) + f(j))^2 \cdot \mu(ij))$ . Instead, we use binary search on the number of removable copies of edges. Assume that we want to remove  $\mu$  copies of edge  $ij \in E(G)$  from the graph. We denote the resulting graph by  $B(G - ij^\mu)$ . In such case  $B(G)$  and  $B(G - ij^\mu)$  differ from one another by  $2\mu$  updates of rank one, i.e.,

$$\begin{aligned} B(G - ij^\mu) &= B(G) + \sum_{k=1}^{\mu} -x^{ij,k}(y^{ij,k})^T + y^{ij,k}(x^{ij,k})^T \\ &= B(G) + [-x^{ij,1}, y^{ij,1}, \dots, -x^{ij,\mu}, y^{ij,\mu}][y^{ij,1}, x^{ij,1}, \dots, y^{ij,\mu}, x^{ij,\mu}]^T. \end{aligned} \tag{32}$$

where  $x^{ij,k}$  and  $y^{ij,k}$  are length  $\phi$  vectors defined in similar way as in the previous section. Observe that the submatrix affected by these changes has size  $2f(i) + 2f(j)$  so, computing  $B(G - ij^\mu)^{-1}$  from  $B(G)^{-1}$  can be realized using [17, Corollary 2.1]. Moreover, we can test non-singularity of  $B(G - ij^\mu)$  by

$$\det(I_\mu + [-x^{ij,1}, \dots, y^{ij,\mu}]^T B(G)^{-1} [y^{ij,1}, \dots, x^{ij,\mu}]) \neq 0. \tag{33}$$

This test requires  $O((f(i) + f(j))^\omega)$  time, because the affected matrix size is  $2f(i) + 2f(j)$ . In the following algorithm we use this test together with a version of binary search.

---

**Algorithm 6** An  $O(m\phi^2)$  time algorithm for finding  $f$ -factor in the multigraph  $G$ .

---

```

1: Let  $B(G)$  be  $\phi \times \phi$  skew-symmetric adjacency matrix of  $G$ 
2: Replace the variables in  $B(G)$  for random elements from  $\mathcal{Z}_p$  for prime  $p = \Theta(\phi^2)$  to obtain  $B$ 
3: (with probability  $\geq 1 - \frac{1}{\phi}$  matrix  $B$  is non-singular when  $B(G)$  is non-singular)  $\triangleright$  by Lemma 1
4: Compute  $B^{-1}$ 
5: (we remove only removable edges so  $B$  remains non-singular during execution of the algorithm)
6: for all  $ij \in E$  do
7:   Let  $\mu$  be the highest power of 2 not higher than  $\min(\mu(ij), f(i), f(j))$ .
8:   Let  $k := 0$ .
9:   while  $\mu \geq 1$  do
10:    if  $\det(I_\mu + [-x^{ij,k+1}, \dots, y^{ij,k+\mu}]^T B^{-1} [y^{ij,k+1}, \dots, x^{ij,k+\mu}]) \neq 0$  then
11:       $\triangleright \mu$  copies of  $ij$  are removable by (33)
12:      Set  $E := E - ij^\mu$ 
13:      Set  $B := B + [-x^{ij,k+1}, \dots, y^{ij,k+\mu}] [y^{ij,k+1}, \dots, x^{ij,k+\mu}]^T$ 
14:       $\triangleright$  This corresponds to  $B(G - ij^\mu)$  by (30)
15:      Recompute  $B^{-1}$   $\triangleright$  Using Sherman-Morrison-Woodbury formula
16:      Set  $k := k + \mu$   $\triangleright$  The number of copies of  $ij$  removed so far
17:    end if
18:    Set  $\mu := \mu/2$   $\triangleright$  The number of edges we try to remove is halved
19:  end while
20: end for
21: Return  $E$   $\triangleright$  All removable copies of edges have been removed, so what remains is an  $f$ -factor

```

---

This time we modify [17, Algorithm 1] in the same way as given by Algorithm 6. The time for updates remains the same, because a submatrix of size  $|S| \times |S|$  is still updated in  $O(|S|^\omega)$  time. On the other hand, the cost we pay to find maximum number  $\mu$  of removable copies of edge  $e$  is  $O((f(i) + f(j))^\omega)$ . The number of removed edges in the binary search forms a geometric series, so the cost is dominated by the first element, which in turn is smaller than the size of the submatrix to power of  $\omega$ . This time we obtain following bounds

$$\begin{aligned}
h(S) &= \sum_i h(S_i) + g(S_1, S_2) + O(|f(S)|^\omega) \\
g(R, S) &= \begin{cases} O(|f(R) + f(S)|^\omega) & \text{if } |R| = |S| = 1, \\ \sum_{i,j} g(R_i, S_j) + O(|f(R) + f(S)|^\omega) & \text{otherwise.} \end{cases}
\end{aligned}$$

The solution for these equations gives an  $O(\phi^\omega)$  time bound.

### 13 Finding perturbed factor weights

This section shows how to compute the quantities  $w(F_v)$  and  $w(F^v)$ , for all  $v \in V$ . (Recall from Section 7 that we are dealing with an  $f$ -critical graph; each  $v \in V$  has a maximum  $f_v$ -factor  $F_v$  and a maximum  $f^v$ -factor  $F^v$ .) We start by considering simple graphs, and then comment on multigraphs. For simplicity assume that the weight function is non-negative, i.e.,  $w : E \rightarrow [0..W]$ . (If this is not the case, redefine  $w(ij) := w(ij) + W$ . This increases the weight of each  $f$ -factor by exactly  $Wf(V)/2$ .)

Following (25), define

$$B(G)_{i,r,j,c} = \begin{cases} z^{w(ij)} x_r^{ij} y_c^{ij} & ij \in E \text{ and } (i, r, j, c) \in \mathbb{P}, \\ -z^{w(ij)} x_c^{ij} y_r^{ij} & ij \in E \text{ and } (j, c, i, r) \in \mathbb{P}, \\ 0 & \text{otherwise,} \end{cases} \quad (34)$$

where  $z$  is a new indeterminate. For the next result we assume  $G$  has an  $f$ -factor. Theorem 47 shows that there is a mapping  $\mathcal{F}$  from terms of  $\det(B(G))$  onto  $\Phi_2(G)$ . The degree of  $z$  in a term  $\sigma$  equals the total weight of the edges used. This gives the following.

**Corollary 49.** *For a simple graph  $G$  that has an  $f$ -factor,  $\deg_z(\det(B(G)))$  is twice the weight of a maximum  $f$ -factor.*

Now suppose  $G$  is  $f$ -critical. For any  $v \in V$  let  $G_v$  be  $G$  with an additional vertex  $t$  joined to  $v \in V$  by a zero weight edge. Set  $f(t) = 1$ . A maximum  $f$ -factor in  $G_v$  weighs the same as a maximum  $f_v$ -factor in  $G$ . For the computation of  $f_v$ -factors we need the following definition. Let  $G_*$  be  $G$  with an additional vertex  $t$  that is connected to all vertices  $v \in V$  with zero weight edges. As previously, we set  $f(t) = 1$ . Let us denote by  $F_*$  the maximum  $f$ -factor in  $G_*$ .

**Lemma 50.**  $\deg_z(\text{adj}(B(G_*))_{v,0,t,0}) = w(F_*) + w(F_v)$ .

*Proof.* Observe that

$$\text{adj}(B(G_*))_{v,0,t,0} = (-1)^{n(t,0)+n(v,0)} \det(B(G_*)^{t,0,v,0}),$$

By Theorem 47 we know that  $\det(B(G_*))$  contains terms corresponding to elements of  $F_2(G_*)$ . Hence, by the above equality, terms of  $\text{adj}(B(G_*))_{v,0,t,0}$  correspond to elements  $F_2(G_*)$  that use edge  $tv$ , but with this edge removed. In other words terms of  $\text{adj}(B(G_*))_{v,0,t,0}$  are obtained by pairing an  $f$ -factor in  $G_*$  and an  $f$ -factor in  $G_v$ , and removing the edge  $tv$ . Similarly, as we observed in Corollary 49 the degree of  $z$  encodes the total weight of elements of  $F_2(G_*)$ . Moreover, the maximum elements are constructed by taking maximum  $f$ -factor of  $G_*$  and maximum  $f$ -factor in  $G_v$ . As we already observed maximum  $f$ -factor in  $G_v$  is an maximum  $f_v$ -factor in  $G^+$  so the theorem follows. ■

The above theorem leads to Algorithm 7 that computes weights of  $F_v$ , for all  $v \in V$ , in  $\tilde{O}(Wn^\omega)$  time.

---

**Algorithm 7** An  $\tilde{O}(W\phi^\omega)$  time algorithm for finding weights of  $F_v$ , for all  $v \in V$ , in a simple graph  $G^+$ .

---

- 1: Let  $B(G_*)$  be  $\phi \times \phi$  matrix representing  $G_*$
  - 2: Replace the variables in  $B(G_*)$  for random elements from  $\mathbb{Z}_p$  for prime  $p = \Theta(\phi^3)$  to obtain  $B$
  - 3: Compute  $d := \det(B)$  ▷ requires  $\tilde{O}(W\phi^\omega)$  time using Theorem 3
  - 4:  $(\deg_z(d) = \deg_z(\det(B(G_*))))$  with probability  $\geq 1 - \frac{1}{\phi^2}$  ▷ by Lemma 1
  - 5: Set  $w(F_*) := \deg_z(d)/2$  ▷ by Corollary 49
  - 6: Compute  $a := \text{adj}(B)_{e_{t,0}} = \det(B)B^{-1}e_{t,0}$  ▷ requires  $\tilde{O}(W\phi^\omega)$  time using Theorem 3
  - 7: **for**  $v \in V$  **do**
  - 8:    $(\deg_z(a_v) = \deg_z(\text{adj}(B(G_*))_{v,0,t,0}))$  with probability  $\geq 1 - \frac{1}{\phi^2}$  ▷ by Lemma 1
  - 9:   Set  $w(F_v) := \deg_z(a_v) - w(F_*)$  ▷ by Lemma 50
  - 10: **end for** ▷ by union bound all  $w(F_v)$  are correct with probability  $\geq 1 - \frac{1}{\phi}$
-

For the computation of  $f^v$ -factors we need to proceed in slightly modified way. We construct  $G^v$  from  $G^+$  by:

- adding new vertices  $t_u$  and a zero weight edges  $ut_u$ , for every vertex  $u \in V$ ,
- adding a new vertex  $t$  and zero weight edge  $tt_v$ .

Moreover, we define  $f'(v) = f(v) + 1$ ,  $f'(t_v) = 1$  and  $f'(t) = 1$ . Finally,  $G^*$  is obtained from  $G^+$  in similar way, but with the difference that  $t$  is connected to all vertices  $t_v$ . Again we observe, that the weight of the maximum  $f'$ -factor in  $G^v$  is equal to the weight of the maximum  $f^v$ -factor in  $G^+$ . This allows us to prove the following.

**Lemma 51.**  $\deg_z(\text{adj}(B(G^*))_{t_v,0,t,0}) = w(F^*) + w(F^v)$ .

*Proof.* By Theorem 47 we know that  $\det(B(G^*))$  contains terms corresponding to elements of  $F_2(G^*)$ . Hence, terms of  $\text{adj}(B(G^*))_{t_v,0,t,0}$  correspond to elements  $F_2(G^*)$  that use edge  $tt_v$ , but with this edge removed. In other words terms of  $\text{adj}(B(G^*))_{t_v,0,v,0}$  are obtained by pairing an  $f'$ -factor in  $G^*$  and an  $f'$ -factor in  $G^v$ . As previously, the maximum weights elements are constructed by taking the maximum  $f'$ -factor of  $G^*$  and the maximum  $f'$ -factor in  $G^v$ . As observed above  $f'$ -factors correspond to  $f^v$ -factors in  $G^+$ . ■

This leads to  $\tilde{O}(Wn^\omega)$  time algorithm for computing weights of  $F^v$ , for all  $v \in V$ .

---

**Algorithm 8** An  $\tilde{O}(W\phi^\omega)$  time algorithm for finding weights of  $F^v$ , for all  $v \in V$ , for a simple graph  $G^+$ .

---

- 1: Let  $B(G^*)$  be  $\phi \times \phi$  matrix representing  $G^*$
  - 2: Replace the variables in  $B(G^*)$  for random elements from  $\mathcal{Z}_p$  for prime  $p = \Theta(\phi^3)$  to obtain  $B$
  - 3: Compute  $d := \det(B)$  ▷ requires  $\tilde{O}(W\phi^\omega)$  time using Theorem 3
  - 4:  $(\deg_z(d) = \deg_z(\det(B(G^*))))$  with probability  $\geq 1 - \frac{1}{\phi^2}$  ▷ by Lemma 1
  - 5: Set  $w(F^*) := \deg_z(d)/2$  ▷ by Corollary 49
  - 6: Compute  $a := \text{adj}(B)e_{t,0} = \det(B)B^{-1}e_{t,0}$  ▷ requires  $\tilde{O}(W\phi^\omega)$  time using Theorem 3
  - 7: **for**  $v \in V$  **do**
  - 8:    $(\deg_z(a_{t_v}) = \deg_z(\text{adj}(B(G^*))_{t_v,0,t,0}))$  with probability  $\geq 1 - \frac{1}{\phi^2}$  ▷ by Lemma 1
  - 9:   Set  $w(F^v) := \deg_z(a_v) - w(F^*)$  ▷ by Lemma 51
  - 10: **end for** ▷ by union bound all  $w(F^v)$  are correct with probability  $\geq 1 - \frac{1}{\phi}$
- 

### 13.1 Multigraphs

Let  $G = (V, E)$  be a multigraph, and let  $w : E \times k \rightarrow \mathbb{Z}$  be the edge weight function. This function assigns weight  $w(e, k)$  to the  $k$ 'th copy of  $e \in E$ . Joining ideas from (34) and (26) we define

$$B(G)_{i,r,j,c} = \begin{cases} \sum_{k=1}^{\mu(ij)} z^{w(ij,k)} x_r^{ij,k} y_c^{ij,k} & ij \in E \text{ and } (i, r, j, c) \in \mathbb{P}, \\ -\sum_{k=1}^{\mu(ij)} z^{w(ij,k)} x_c^{ij,k} y_r^{ij,k} & ij \in E \text{ and } (j, c, i, r) \in \mathbb{P}, \\ 0 & \text{otherwise,} \end{cases}$$

where  $z$  is a new indeterminate. Theorem 47 shows that there is a mapping  $\mathcal{F}$  from terms of  $\det(B(G))$  onto  $\mathcal{F}_2(G)$ . Observe that the construction from previous section requires only the existence of such mapping, so it can be used for multigraphs as well.

## 14 Conclusions and open problems

This paper presents new algebraic algorithms for the fundamental problems of  $b$ -matching, undirected single-source shortest paths, and  $f$ -factors. Some intriguing open problem and challenges emerge from this study:

- The matrices we construct for unweighted  $f$ -factors have a very special block structure. Can this block structure be exploited to obtain faster algorithms, e.g., time  $O(\phi^{\omega-1}n)$ ?
- Can the running time of our algebraic max-flow algorithm be improved, perhaps by combining it with scaling techniques? Can scaling be used in the non-bipartite algorithms?
- We gave first algebraic algorithms for simple 2-factors. Are there algebraic formulations for triangle-free or square-free 2-factors? If so one expects the resulting algorithms to be simpler than existing combinatoric ones.
- What is the complexity of all-pairs undirected shortest distances on conservative graphs? Can  $\tilde{O}(Wn^\omega)$  time be achieved, as in the case of non-negative weights?

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## A Allowed edges

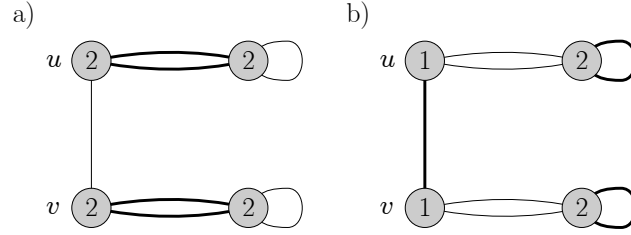


Figure 8: The only 2-factor is shown on panel a), whereas the only  $f_{u,v}$ -factor is shown on panel b). Edge  $uv$  is not allowed in any 2-factor, although  $f_{u,v}$ -factor does exist.

Observe that for any edge  $ij$  as long as  $f(i) = f(j) = 1$  the existence of  $f_{i,j}$ -factor is equivalent to the fact that  $ij$  is allowed. However, as shown on Figure 8, the edge  $uv$  is not allowed, although the  $f_{u,v}$ -factor does exist. The reason for this is that we are trying to use edge  $uv$  twice. Hence, similar criteria as in Corollary 9 does not hold in non-bipartite case.